BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2011

M1M1 (January Test)

Mathematical Methods I

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. The function $f(x) = \log(1 + \tan x)$ is defined for $a < x < b$, where $b > 0 > a$.

   (a) What is the largest possible value of $b - a$? From now on, we assume a and b take these maximal values.

   (b) Find the first 2 non-zero terms in the power series for $f(x)$ about $x = 0$.

   (c) What do you expect will be the Radius of Convergence of the Maclaurin series for $f(x)$?

   (d) Draw a rough sketch of $f(x)$ over the interval (a, b).

   (e) Show that $f'(x) < 1$ for $0 < x < \frac{1}{4}\pi$.

   (f) Use the Mean Value Theorem over two separate ranges to show that

$$\log(2) - \tfrac{1}{8}\pi < f(\tfrac{1}{8}\pi) < \tfrac{1}{8}\pi.$$

   (g) Discuss whether or not the integral

$$\int_{-\pi/4}^{\pi/4} f(x)\,dx \qquad \text{exists.}$$

SECTION B

2.

(a) (i) If a and b are real constants and n is a positive integer, use complex numbers to show that

$$\left(\frac{d}{dx}\right)^n [e^{ax}\sin(bx)] = (a^2+b^2)^{n/2} e^{ax}\sin(bx+\phi)$$

where you should define ϕ.

(ii) Is there any sense in substituting $n = -1$ in the above formula?

(b) If $\tan z = t$, show that

$$e^{2iz} = \frac{1+it}{1-it}.$$

Hence show that if t is a real number, then z must also be real.

3.

(a) If $a(x)$ and $b(x)$ are given functions, find the general solution of the ODE for $y(x)$,

$$y' - \frac{a'y}{a} = b.$$

Hence find the solution when

$$a(x) = (x+1)\sin x \qquad \text{and} \quad b(x) = \frac{\sin x}{(x-1)}$$

(b) Derive an expression for the second derivative of x with respect to y in terms of the first two derivatives of y with respect to x.

[**Note:** *x and y in part (b) are not the same as in part (a)*

Solutions

1.(a) $1+\tan x$ is infinite at $x=\frac{1}{2}\pi$, and zero at $x=-\frac{1}{4}\pi$, so the logarithm becomes infinite at these points. So the largest possible value of $b-a$ is $\frac{3}{4}\pi$. **1 mark**

(b) $\log(1+\tan x) = \tan x - \frac{1}{2}\tan^2 x + \ldots$ and $\tan x = x + O(x^3)$. Thus $\log(1+\tan x) = x - \frac{1}{2}x^2 + O(x^3)$. Alternatively use a Taylor series. **2 marks**

(c) $f(x)$ is continuous and differentiable for $-\frac{1}{4}\pi < x < \frac{1}{2}\pi$, but is singular at the endpoints. The nearest singular point is $\frac{1}{4}\pi$ away from the origin, so the radius of convergence is $\frac{1}{4}\pi$.**2 marks**

(d) See below. $f'(x) = \frac{\sec^2 x}{1+\tan x} > 0$ over the range of interest, so curve increases. **3 marks**

(e) Now $0 < \tan x < 1$ for $0 < x < \frac{1}{4}\pi$. Thus $\tan^2 x < \tan x$ and

$$f'(x) = \frac{\sec^2 x}{1+\tan x} = \frac{1+\tan^2 x}{1+\tan x} < \frac{1+\tan x}{1+\tan x} = 1$$ **4 marks**

(f) Over $(0, \frac{1}{8}\pi)$, the MVT states

$$f(\tfrac{1}{8}\pi) - f(0) = (\tfrac{1}{8}\pi - 0)f'(\xi) < \tfrac{1}{8}\pi \quad \text{by part(e)}$$

for some ξ between 0 and $\frac{1}{8}\pi$. Over the range $(\frac{1}{8}\pi, \frac{1}{4}\pi)$, we have

$$f(\tfrac{1}{4}\pi) - f(\tfrac{1}{8}\pi) = \tfrac{1}{8}\pi f'(\eta) < \tfrac{1}{8}\pi$$

for some η in $(\frac{1}{8}\pi, \frac{1}{4}\pi)$. Now $f(0) = 0$ and $f(\frac{1}{4}\pi) = \log 2$. Putting these together,

$$\log(2) - \tfrac{1}{8}\pi < f(\tfrac{1}{8}\pi) < \tfrac{1}{8}\pi.$$ **5 marks**

(g) The integrand is continuous except at the endpoint $x = -\frac{1}{4}\pi$. Near this singular point, writing $x = -\frac{1}{4}\pi + t$, we have the Taylor series $\tan x = \tan(-\frac{1}{4}\pi) + \sec^2(-\frac{1}{4}\pi)t + O(t^2)$ Thus $1 + \tan x \simeq 2t + O(t^2)$. Thus $f(x) \simeq \log(t)$ near $t = 0$. $\int \log t\, dt = t\log t - t$ is zero at $t = 0$, so the singularity is integrable. We conclude the integral exists. **3 marks**

2.

(a) $e^{ax}\sin(bx) = \Im m\left[e^{(a+ib)x}\right]$. Thus

$$\left(\frac{d}{dx}\right)^n [e^{ax}\sin(bx)] = \Im m\left[(a+ib)^n e^{(a+ib)x}\right].$$

Writing $a+ib = re^{i\alpha}$, we have $r = \sqrt{a^2+b^2}$ and $\cos\alpha = a/r$, $\sin\alpha = b/r$. Then $(a+ib)^n = r^n e^{in\alpha}$. So

$$\left(\frac{d}{dx}\right)^n [e^{ax}\sin(bx)] = \Im m\left[r^n e^{ax} e^{i(bx+n\alpha)}\right] = (a^2+b^2)^{n/2} e^{ax}\sin(bx+n\alpha)$$ **8 marks**

as required, with $\phi = n\alpha$ **2 marks**.

Putting $n = -1$ in the RHS is the equivalent of sending $e^{(a+ib)x}$ to $\frac{1}{a+ib}e^{(a+ib)x}$, which is performing an integration. So the formula still holds if we interpret $(d/dx)^{-1}$ as an anti-derivative or integral, and if we remember an arbitrary constant. **2 marks**

(b) We have

$$t = \tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz} - e^{-iz})/2i}{(e^{iz} + e^{-iz})/2} = \frac{e^{2iz} - 1}{i(e^{2iz} + 1)}.$$

Solving for e^{2iz}, we have

$$e^{2iz} = \frac{1 + it}{1 - it}$$ **4 marks**

Now for any complex number ζ, $\log \zeta = \log|\zeta| + i \arg(\zeta) + 2k\pi i$. If $\zeta = (1 + it)/(1 - it)$, then $|\zeta| = 1$. Taking the logarithm, we have $2iz = \log 1 + i(\arg \zeta + 2k\pi)$ and so z is real as required.**4 marks**

3. The ODE is linear, and so can be solved by an integrating factor.

$$I = \exp(\int \frac{-a'}{a} dx) = \exp(-\log a) = a^{-1}.$$ **2 marks**

Multiplying by a^{-1}, we have

$$\frac{y'}{a} - \frac{a'}{a^2} = \frac{b}{a}$$

Or $(y/a)' = b/a$, and hence

$$y(x) = a(x) \int^x \frac{b(t)}{a(t)}\, dt + Ca(x)$$

is the general solution. (Even though a constant is implied by the indefinite integral, deduct 2 marks unless it is imade clear that the constant multiplies $a(x)$) **6 marks**

Thus if $a(x) = (x + 1)\sin x$ and $b(x) = \sin x/(x - 1)$, we need the integral

$$\int \frac{\sin x/(x-1)}{(x+1)\sin x} dx = \int \frac{dx}{x^2 - 1} = \tfrac{1}{2} \log\left|\frac{x-1}{x+1}\right|.$$ **3 marks**

using partial fractions. (Also accept $\tanh^{-1}$)

So the solution is

$$y = \tfrac{1}{2}(x + 1)\sin x \left(\log\left|\frac{x-1}{x+1}\right| + C\right)$$ **4 marks**.

(b) (This was on a problem sheet). We have $dx/dy = 1/(dy/dx)$. Thus

$$\frac{d^2x}{dy^2} = -\frac{1}{(dy/dx)^2}\frac{d}{dy}\left(\frac{dy}{dx}\right) = -\frac{1}{(dy/dx)^2}\frac{d^2y}{dx^2}\frac{dx}{dy} = -\frac{1}{(dy/dx)^3}\frac{d^2y}{dx^2}$$ **5 marks**