Solutions to 2010 January Test

1.(a) Write $y = (x/(2-x))^a$. Then provided $a \neq 0$ we have

$$y^{1/a} = \frac{x}{2-x} \qquad \text{or} \quad x = \frac{2y^{1/a}}{1+y^{1/a}}.$$

Thus the inverse function exists provided $a \neq 0$ **[1 marks]**

$$\text{and is} \qquad g(x) \equiv f^{-1}(x) = \frac{2x^{1/a}}{1+x^{1/a}}.$$ **[2 marks]**

Note if $a = 0$ then $f(x) = 1$ for all x and is obviously noninvertible.

(b) We have $f(x) = f(1) + (x-1)f'(1) + \frac{1}{2}(x-1)^2 f''(1) + \ldots$ and $f(1) = 1$. Now

$$f'(x) = a\left(\frac{x}{2-x}\right)^{a-1}\left(\frac{2}{(2-x)^2}\right) = 2a\frac{x^{a-1}}{(2-x)^{a+1}} \qquad \Longrightarrow \quad f'(1) = 2a.$$

Differentiating again,

$$f''(x) = 2a(a-1)x^{a-2}(2-x)^{-1-a} + 2a(a+1)x^{a-1}(2-x)^{-a-2} \qquad \Longrightarrow \quad f''(1) = 4a^2.$$

Putting all this together, the first 3 terms are

$$f(x) = 1 + 2a(x-1) + 2a^2(x-1)^2 + \ldots$$ **[4 marks]**

(c) Now

$$f(1+e^{i\theta}) = \left[\frac{1+e^{i\theta}}{1-e^{i\theta}}\right]^a = \left[\frac{e^{-i\theta/2}+e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}}\right]^a = \left[\frac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta}\right]^a = (\cot\tfrac{1}{2}\theta)^a e^{i\pi a/2}.$$

[4 marks]

Note for θ in the given range $\cot\theta/2 > 0$, and as a is an integer $e^{2\pi i a} = 1$.

(d) See sketch **[3 marks]**

(e) As $a \to \infty$, $t^a \to 0$ if $|t| < 1$, and has no finite limit if $|t| > 1$. Now as $0 < x < 2$ we have $|x/(2-x)| < 1$ for $x < 2-x$ or $0 < x < 1$. Furthermore, $f(1) = 1$ for all values of a. Therefore

$$f_\infty(x) = \begin{cases} 0 & \text{for} \quad 0 < x < 1 \\ 1 & \text{for} \quad x = 1 \\ \text{undefined} & \text{for} \quad 1 < x < 2. \end{cases}$$ **[2 marks]**

Clearly $f_\infty(x)$ is not continuous at $x = 1$, although $f_\infty(1)$ is defined. **[1 marks]**

(f) The integrand is continuous except possibly at $x = 0$ and $x = 2$. Near $x = 0$, the integral exists provided $a > -1$. Near $x = 2$, the integral exists provided $a < 1$. Combining these constraints, the integral exists provided $-1 < a < 1$. **[3 marks]**

[Total 20]

2.(a) substituting $\theta = 0$, we see $P_n(1) = 1$. Taking the limits as $\theta \to 0$, $\sin(n\theta)/\sin\theta \to n$. Thus $Q_{n-1}(1) = n$ and so $Q_n(1) = n + 1$. **[2 marks]**

(b) Now $\cos(n\theta + \theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta$ and so

$$P_{n+1}(\cos\theta) = P_n(\cos\theta)\cos\theta - \sin^2\theta\, Q_{n-1}(\cos\theta). \qquad \textbf{[3 marks]}$$

Substituting $x = \cos\theta$, we have

$$P_{n+1}(x) = xP_n(x) - (1 - x^2)Q_{n-1}(x),$$

as required. Similarly, using $\sin(n\theta + \theta) = \sin n\theta \cos\theta + \cos n\theta \sin\theta$,

$$\sin\theta\, Q_n(\cos\theta) = \sin\theta\, Q_{n-1}(\cos\theta)\cos\theta + P_n(\cos\theta)\sin\theta$$

or

$$Q_n(x) = xQ_{n-1}(x) + P_n(x). \qquad \textbf{[3 marks]}$$

(c) Replacing n by $n + 1$ in this last formula (2) and then using the first (1), we have

$$Q_{n+1}(x) = xQ_n(x) + xP_n(x) - (1 - x^2)Q_{n-1}(x) = xQ_n + x(Q_n - xQ_{n-1}) - (1 - x^2)Q_{n-1},$$

using the 2nd formula again. Rearranging,

$$Q_{n+1}(x) - 2xQ_n(x) + Q_{n-1}(x) = 0. \qquad \textbf{[3 marks]}$$

Similarly, replacing n by $n + 1$ in the first formula and using the second, we have

$$P_{n+2} = xP_{n+1} - (1 - x^2)[P_n + xQ_{n-1}] = xP_{n+1} - (1 - x^2)P_n - x(xP_n - P_{n+1})$$

using the first again. Rearranging, we have

$$P_{n+2} = 2xP_{n+1} - P_n \qquad \Longrightarrow \qquad P_{n+1}(x) - 2xP_n(x) + P_{n-1}(x) = 0, \qquad \textbf{[3 marks]}$$

replacing $n + 1$ by n.

(d) Differentiating with respect to θ, we have

$$-n\sin n\theta = P_n'(\cos\theta)(-\sin\theta) \qquad \Longrightarrow \qquad P_n'(x) = nQ_{n-1}(x). \qquad \textbf{[2 marks]}$$

and

$$n\cos n\theta = \cos\theta\, Q_{n-1}(\cos\theta) - \sin^2\theta Q_{n-1}'(\cos\theta)$$

or

$$nP_n(x) = xQ_{n-1}(x) - (1 - x^2)Q_{n-1}'(x). \qquad \textbf{[2 marks]}$$

Differentiating with respect to x, we have

$$P_n''(x) = nQ_{n-1}'(x) \qquad \Longrightarrow \qquad (1 - x^2)P_n''(x) = n[xQ_{n-1}(x) - nP_n(x)] = xP_n'(x) - n^2P_n(x)$$

using the above. Rearranging, we have

$$(1-x^2)P_n''(x) - xP_n'(x) + n^2 P_n(x). \qquad \textbf{[2 marks]}$$

[Total 20]

3.(a) If $f(x)$ is continuous for $a \leqslant x \leqslant b$ and differentiable for $a < x < b$, then there exists a ξ in $a < \xi < b$ such that

$$f(b) - f(a) = (b-a)f'(\xi) \qquad \textbf{[2 marks]}$$

Choose $f(x) = \log x$, so that $f'(x) = 1/x$. Then there is a ξ with $b > \xi > a$ such that

$$\log b - \log a = \frac{b-a}{\xi} \qquad \Longrightarrow \quad \xi = \frac{b-a}{\log(b/a)}.$$

As ξ is between a and b we therefore have as required

$$b > \frac{b-a}{\log(b/a)} > a. \qquad \textbf{[5 marks]}$$

Now replace b by x and choose $a = 1$. Then

$$x > \frac{x-1}{\log x} > 1 \qquad \Longrightarrow \quad \int_1^2 x\,dx > \int_1^2 \frac{x-1}{\log x}\,dx > \int_1^2 dx$$

Or

$$\tfrac{3}{2} > \int_1^2 \frac{x-1}{\log x}\,dx > 1 \qquad \textbf{[4 marks]}$$

(b) Equation is linear, rewrite as $y' - y = 3x^2 - x^3$ when the integrating factor is e^{-x}. Then

$$(ye^{-x})' = (3x^2 - x^3)e^{-x} \qquad \Longrightarrow \quad ye^{-x} = \int (3x^2 - x^3)e^{-x}\,dx \qquad \textbf{[3 marks]}$$

Integrating by parts

$$\int (3x^2 - x^3)e^{-x}\,dx = (x^3 - 3x^2)e^{-x} + \int (6x - 3x^2)e^{-x}\,dx = x^3 e^{-x} + c$$

after some more algebra. So the general solution is

$$y = Ae^{-x} + x^3 \qquad \textbf{[4 marks]}$$

for an arbitrary constant A.

(c) Both the numerator and denominator are zero when $x = 1$, so use de l'Hôpital's rule.

$$\lim_{x\to 1}\left[\frac{\sin \pi x + \cos \pi x + 1}{1 - x^3 + \log x}\right] = \lim_{x\to 1}\left[\frac{\pi\cos \pi x - \pi \sin \pi x}{-3x^2 + 1/x}\right] = \tfrac{1}{2}\pi \qquad \textbf{[2 marks]}$$

[Total 20]