SECTION A

1. The function $f(x)$ is defined in terms of a real parameter a by

$$f(x) = \left(\frac{x}{2-x}\right)^a \qquad \text{for } 0 < x < 2. \qquad (*)$$

(a) Find the inverse function $g(x) \equiv f^{-1}(x)$ and state for which values of a the inverse exists.

(b) Find the first three non-zero terms of the power series of $f(x)$ about $x = 1$.

(c) The formula $(*)$ is extended to apply also for complex values of x. Express in polar form the complex number $f(1 + e^{i\theta})$ where $0 < \theta < \pi$ and a is a positive integer.

(d) When $a = \frac{1}{2}$, sketch the function $g(x)$ defined in part (a) over its domain.

(e) The function $f_\infty(x)$ is defined for fixed x in $0 < x < 2$ as the limit

$$f_\infty(x) = \lim_{a\to\infty} [f(x)].$$

Evaluate $f_\infty(x)$ when the limit exists and identify a value x_0 such that $f_\infty(x_0)$ exists but $f_\infty(x)$ is not continuous at $x = x_0$.

(f) For which values of a does the integral

$$\int_0^2 f(x)\, dx \qquad \text{exist?}$$

SECTION B

2.

(a) State carefully the Mean Value Theorem for a function $f(x)$ defined in $a \leqslant x \leqslant b$.

Use the theorem to prove that provided $b > a > 0$, then

$$b > \frac{b-a}{\log(b/a)} > a.$$

Deduce that

$$\frac{3}{2} > \int_1^2 \frac{x-1}{\log x}\, dx > 1.$$

(b) Find the general solution of the ODE

$$\frac{dy}{dx} = y + 3x^2 - x^3.$$

(c) Evaluate the limit

$$\lim_{x\to 1}\left[\frac{\sin \pi x + \cos \pi x + 1}{1 - x^3 + \log x}\right].$$

3. The functions $\sin n\theta$ and $\cos n\theta$, where n is a positive integer, can be expressed as

$$\cos n\theta = P_n(\cos\theta) \qquad \sin n\theta = \sin\theta\, Q_{n-1}(\cos\theta) \tag{0}$$

where $P_n(x)$ and $Q_n(x)$ are polynomials of degree n.

(a) What must be the values of $P_n(1)$ and $Q_n(1)$?

(b) Show that

$$P_{n+1}(x) = xP_n(x) - (1-x^2)Q_{n-1}(x) \tag{1}$$

and

$$Q_n(x) = P_n(x) + xQ_{n-1}(x). \tag{2}$$

(c) Deduce from (1) and (2) that

$$Q_{n+1}(x) - 2xQ_n(x) + Q_{n-1}(x) = 0 \tag{3}$$

and

$$P_{n+1}(x) - 2xP_n(x) + P_{n-1}(x) = 0. \tag{4}$$

(d) Show further that if $'$ denotes differentiation, then

$$P_n'(x) = nQ_{n-1}(x) \tag{5}$$

and

$$(1-x^2)Q_{n-1}'(x) = xQ_{n-1}(x) - nP_n(x). \tag{6}$$

Deduce that

$$(1-x^2)P_n''(x) - xP_n'(x) + n^2P_n(x) = 0. \tag{7}$$