Solutions to 2009 January Test

1.(a) $f = x\exp(-x^2)$ so $f'(x) = (1-2x^2)\exp(-x^2)$ and $f''(x) = (-6x+4x^3)\exp(-x^2)$. Stationary points at $x = \pm 1/\sqrt{2}$, $f''(1/\sqrt{2}) < 0$ giving a maximum at $(1/\sqrt{2},\ e^{-1/2}/\sqrt{2})$. As $f(x)$ is odd there is a minimum at $(-1/\sqrt{2},\ -e^{-1/2}/\sqrt{2})$. Inflection points where $f'' = 0$ i.e. when $x = 0,\ \pm\sqrt{3/2}$. So inflection at $(0, 0)$, $(\pm\sqrt{3/2},\ \pm\sqrt{3/2}e^{-3/2})$. **[2 marks]**

(b) See below **[2 marks]**

(c) As graph has a maximum, there are two positive values of x giving the same y value for $0 < y < e^{-1/2}/\sqrt{2}$. As $y \to 0$ these two values tend to $x = 0$ and $x \to \infty$. Thus the difference between the two values tends to zero as $y \to e^{-1/2}$ and tends to infinity as $y \to 0$. Hence by continuity the difference will be 2π somewhere, in fact for $x \simeq 2\pi\exp(-4\pi^2)$ (not required). **[2 marks]**

(d) $f(a+ib) = (a+ib)\exp(-a^2+b^2-2abi) = \exp(b^2-a^2)(a+ib)(\cos 2ab - i\sin 2ab)$ Hence the real part is

$$\Re e[f(a+ib)] = \exp[b^2-a^2](a\cos 2ab + b\sin 2ab) \qquad \textbf{[3 marks]}$$

(e) We have that for some ξ such that $-1 < \xi < x$,

$$f(x) = f(-1) + f'(-1)(x+1) + \tfrac{1}{2}f''(\xi)(x+1)^2.$$

From part (a) $f(-1) = -1/e$, $f'(-1) = -1/e$ and $f''(\xi) = (4\xi^3 - 6\xi)\exp(-\xi^2)$. Hence

$$f(x) = \frac{-1}{e} - \frac{1}{e}(x+1) + (2\xi^3 - 3\xi)\exp(-\xi^2)(x+1)^2. \qquad \textbf{[3 marks]}$$

(f) The integrand is finite everywhere and tends to zero exponentially as $x \to \infty$, so integral exists. Substituting $u = x^2$, $du = 2xdx$

$$\int_0^\infty x^3 e^{-x^2}\,dx = \int_0^\infty \tfrac{1}{2}ue^{-u}\,du = \left[-\tfrac{1}{2}ue^{-u}\right]_0^\infty + \tfrac{1}{2}\int_0^\infty e^{-u}\,du = \tfrac{1}{2}. \qquad \textbf{[2 marks]}$$

(g) ODE is linear. The integrating factor is $\exp\left(-\int 2x\,dx\right) = \exp(-x^2)$. So

$$\left(ye^{-x^2}\right)' = xe^{-x^2} \implies ye^{-x^2} = \int xe^{-x^2} = -\tfrac{1}{2}e^{-x^2} + C \implies y = -\tfrac{1}{2} + Ce^{x^2} \quad \textbf{[3 marks]}$$

(h) Using Leibniz' formula, and noting that $(\exp(-x^2))' = -2f(x)$ we have

$$\left(xe^{-x^2}\right)^{(n)} = x\left(e^{-x^2}\right)^{(n)} + n\left(e^{-x^2}\right)^{(n-1)} = -2xf^{(n-1)} - 2nf^{(n-2)}. \qquad \textbf{[2 marks]}$$

Thus $f^{(n)}(0) = -2nf^{(n-2)}(0)$. Using the ratio test, we find the Maclaurin series converges for all x

$$\lim_{n\to\infty}\left|\frac{f^{(n)}(0)x^n/n!}{f^{(n-2)}(0)x^{n-2}/(n-2)!}\right| = \lim_{n\to\infty}\left|\frac{x^2(-2n)}{n(n-1)}\right| = 0 < 1. \qquad \textbf{[1 marks]}$$

2.(i) Using the Mean Value Theorem for the function $f(x) = -\log(1-x)$ on $(0, x)$

$$\frac{-\log(1-x)+\log 1}{x-0} = f'(\xi) = \frac{1}{1-\xi} \qquad \text{for some } \xi \text{ in } 0<\xi<x<1.$$

Now $1/(1-\xi)$ is an increasing function and so $1 < (1-\xi)^{-1} < (1-x)^{-1}$. Thus

$$1 < -\frac{\log(1-x)}{x} < \frac{1}{1-x} \qquad \Longrightarrow \qquad x < -\log(1-x) < \frac{x}{1-x}$$ [**6 marks**]

as required. Now if we assume the series expansions

$$\log(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \ldots \qquad \text{and} \qquad \frac{x}{1-x} = x(1+x+x^2+x^3+\ldots)$$

we see that

$$x < x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \ldots < x + x^2 + x^3 + \ldots$$

which is obviously true as all terms are positive. [**3 marks**]

(ii) Taking the logarithm, we have

$$\lim_{n\to\infty} \log[\eta(n)] = \lim_{n\to\infty} n\log\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} \left(n(-1/n + O(1/n^2))\right) = -1$$

Thus $\eta_\infty = e^{-1}$. [**4 marks**]

Now writing $x = \frac{1}{60}$ in part (i) we have

$$\frac{1}{60} < -\log(1-\frac{1}{60}) < \frac{\frac{1}{60}}{1-\frac{1}{60}} \qquad \Longrightarrow \qquad 1 < -\log[\eta(60)] < \frac{60}{59}$$

Thus

$$e^{-1} > \eta(60) > e^{-60/59} \qquad \Longrightarrow \qquad 1 > \frac{\eta(60)}{\eta_\infty} > e^{-1/59}$$ [**5 marks**]

Also from part (i) we have

$$e^{-x} > 1-x \qquad \Longrightarrow \qquad e^{-1/59} > \frac{58}{59}$$ [**2 marks**]

and the result follows.

3. We have

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d}{d\theta}\left(\frac{1}{r}\right).$$ [**2 marks**]

Differentiating again,

$$-\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d^2r}{dt^2}\frac{dt}{d\theta} = r^2\frac{d^2r}{dt^2} = r^2\left(-\frac{1}{r^2}+\frac{r}{r^4}\right)$$

or writing $u = 1/r$,

$$\frac{d^2u}{d\theta^2} + u = 1. \qquad \textbf{[5 marks]}$$

Substituting $u = 1 + e\sin(\theta - \alpha)$ we get $-e\sin(\theta-\alpha) + 1 + e\sin(\theta - \alpha) = 1$ so this is a solution. As it has two arbitrary constants, it is the general solution of a 2nd order ODE. **[3 marks]**

As θ varies, the sine oscillates between ± 1. Thus r will oscillate between $1/(1-e)$ and $1/(1+e)$ provided $|e| < 1$, and the curve will close on itself as r is 2π-periodic in θ. If $|e| \geqslant 1$, then r heads off to infinity when $\sin(\theta - \alpha) = -1/e$. When $r < 0$ there is no curve. We see there are two types of possible solutions, bounded and unbounded. **[3 marks]**

When $e = \sqrt{2}$ and $\alpha = 0$, we have $r \to \infty$ when $\sin\theta = 1/\sqrt{2}$, or when $\theta = \frac{1}{4}\pi$ and $\theta = \frac{3}{4}\pi$. r is positive for $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$.

Translating to Cartesians, we have

$$1 = r + er\sin\theta = \sqrt{x^2 + y^2} + y\sqrt{2}.$$

Thus

$$1 + 2y^2 - 2\sqrt{2}y = x^2 + y^2 \qquad \Longrightarrow \qquad (y - \sqrt{2})^2 - x^2 = 1$$

which is a rectangular hyperbola. Only one branch appears in the polar curve – squaring has introduced a spurious solution. See below.

When $e = 1/\sqrt{2}$, we have

$$1 - y\sqrt{2} + \tfrac{1}{2}y^2 = x^2 + y^2 \qquad \Longrightarrow \qquad x^2 + \tfrac{1}{2}(y + \sqrt{2})^2 = 1$$

which is an ellipse. From the polar form of the equation it is easy to see that the maximum values of r occurs when $\theta = -\frac{1}{2}\pi$ and the minimum at $\theta = \frac{1}{2}\pi$.