Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2009

M1M1 (January Test)

Mathematical Methods I

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. Define the function $f(x)$ to be

$$f(x) = xe^{-x^2}.$$

(a) Find all stationary points and points of inflection of the graph $y = f(x)$. **[2 marks]**

(b) Sketch the graph in part (a) indicating any features of interest. **[2 marks]**

(c) Is there a value of x for which $f(x) = f(x + 2\pi)$? Justify your answer. **[2 marks]**

(d) If the definition of $f(x)$ holds for complex x values, find the real part of $f(a + ib)$ where a and b are real. **[3 marks]**

(e) Find the first two terms of the Taylor series of $f(x)$ about the point $x = -1$. Assuming $x > -1$ give the exact form of the error term. **[3 marks]**

(f) Does the infinite integral below exist? If so, evaluate it.

$$\int_0^\infty x^2 f(x)\, dx$$ **[2 marks]**

(g) Find the general solution of the ODE

$$y' - 2xy = x.$$ **[3 marks]**

(h) If $f^{(n)}$ denotes the n'th derivative of f, show that for $n \geqslant 2$

$$f^{(n)} = -2nf^{(n-2)} - 2xf^{(n-1)}.$$

Deduce that the Maclaurin series for $f(x)$ converges for all values of x. **[3 marks]**

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SECTION B

2. (i) Use the Mean Value Theorem to prove that

$$\frac{x}{1-x} > -\log(1-x) > x \qquad \text{for } 0 < x < 1.$$ **[6 marks]**

Write down the power series expansions of the first two expressions and check that they are consistent with the inequality. **[3 marks]**

(ii) A Jewish mystical belief is that to visit a sick person removes $1/60$ of his or her pain. Some people erroneously inferred from this that a person would be completely cured if they received 60 visitors. In fact, after 60 visits, the person's pain should logically be reduced to a proportion $\eta(60)$ of its original level where the function

$$\eta(n) = \left(1 - \frac{1}{n}\right)^n.$$

Evaluate $\eta_\infty = \lim_{n\to\infty} \eta(n)$. **[4 marks]**

Using part (i), show that

$$1 > \frac{\eta(60)}{\eta_\infty} > e^{-1/59} > \frac{58}{59}.$$ **[7 marks]**

3. Polar coordinates (r, θ) are defined so that $x = r\cos\theta$ and $y = r\sin\theta$ with $r \geqslant 0$. The path of a spaceship is given parametrically in polar coordinates centred on the sun by $(r(t), \theta(t))$ where t is time. The governing equations can be shown to be

$$r^2\frac{d\theta}{dt} = 1, \qquad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{1}{r^2}.$$

Show that

$$\frac{dr}{dt} = -\frac{d}{d\theta}\left(\frac{1}{r}\right),$$ **[2 marks]**

and deduce that

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) = 1.$$ **[5 marks]**

Verify that

$$\frac{1}{r} = 1 + e\sin(\theta - \alpha) \qquad \text{for arbitrary } e \text{ and } \alpha$$

is a solution to this second order ODE, and explain why it is the general solution to the ODE. **[3 marks]**

Discuss how r behaves as θ varies and identify two different types of solution. **[2 marks]**

Plot the solution curves in the (x, y)-plane when $\alpha = 0$ for the two cases (i) $e = \sqrt{2}$ and (ii) $e = 1/\sqrt{2}$. **[8 marks]**