BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2007

M1M1 (Test)

Mathematical Methods 1

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (a) Show that if the function $x(t)$ is twice-differentiable, and if $v = dx/dt$, then

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

(b) If $x(t)$ obeys the 2nd order differential equation

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{1}{x}\left(\frac{dx}{dt}\right)^2,$$

find all possible possible functions $v(x)$, where v is as in part (a).

(c) If additionally it is known that at $t = 0$, $x = 1$ and $v = 1$ deduce that

$$x = \exp(e^t - 1) \equiv f(t).$$

(d) What is the range of the function $f(t)$ in part (c) if t can take all real values? What is the inverse function $f^{-1}(x)$? What is the domain of the inverse function?

(e) Locate any stationary point or points of inflection of the curve $x = f(t)$.

(f) Find the first 3 terms in the Maclaurin series for $f(t)$.

(g) Calculate the limit

$$\lim_{t\to 0}\left[\frac{\log f(t)}{(f(t)-1)^{2/3}}\right].$$

(h) If the formula for $f(t)$ in (c) holds for complex values of t, find $\Re e(x)$ when $t = 2i$.

SECTION B

2. (a) State the Mean Value Theorem precisely.

The functions $f(x)$ and $g(x)$ are differentiable everywhere, and the derivatives $f'(x)$ and $g'(x)$ are continuous at $x = a$.

Use the Mean Value Theorem to prove that if $f(a) = 0 = g(a)$ and $g'(a) \neq 0$, then

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(a)}{g'(a)}.$$

(b) Suppose $f(x)$ and $g(x)$ are differentiable functions with $f(a) = 0 = g(a)$.
De l'Hôpital's rule states that IF the limit

$$\lim_{x \to a} \left[\frac{f'(x)}{g'(x)} \right]$$

exists THEN

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to a} \left[\frac{f'(x)}{g'(x)} \right].$$

Discuss carefully the use of de l'Hôpital's rule for the two cases:

(i) $\quad f(x) = \log(\sin(\pi x)), \qquad g(x) = (2x-1)^2, \qquad a = \frac{1}{2}.$

(ii) $\quad f(x) = x^3 \sin\left(\frac{1}{x}\right), \qquad g(x) = 1 - \cos x, \qquad a = 0.$

If possible, evaluate both limits.

3. (a) Find a formula for the n'th derivative of the function

$$f(x) = \log(x + a),$$

where a is a positive real number. Hence derive the power series of $f(x)$ about $x = 0$. What is the radius of convergence of the series?

(b) Assume the series you have found is valid when $a = i$ with x remaining real. By considering the imaginary part of the complex logarithm, deduce that for $x > 0$

$$\sin^{-1}\left[\frac{1}{\sqrt{x^2+1}} \right] = \tfrac{1}{2}\pi - x + \tfrac{1}{3}x^3 + \ldots$$

and give the general term in the series.

(c) Calculate $g'(x)$ where

$$g(x) = \sin^{-1}\left[\frac{1}{\sqrt{x^2+1}} \right]$$

and verify that it equals the derivative of the series in part (b).