SECTION A

1.(a) Let

$$f(x) = \frac{x^2+2}{x^2-1}$$

Put $f(x)$ into partial fraction form. Sketch a graph of $f(x)$, carefully labelling all the important features of your graph. Find the first three non-zero terms in the Taylor series of $f(x)$ about $x = 0$.

(b) Find the derivative of $e^{\sqrt{x}}$ from first principles.

Using any method, find expressions for the derivative of $y(x)$ when

$$y(x) = \log\cos x; \quad y(x) = (1+x)^x; \quad y(x) = \int_0^{x^2} \log(1+x'^2)dx'.$$

(c) Find the general solution of the following ordinary differential equations:

$$\frac{dy}{dx} + 2xy = x; \quad x\frac{dy}{dx} + y = \frac{y^2}{x}.$$

(d) Find all complex solutions z of the following equations and sketch the solutions in the complex plane:

(i)

$$z^2\bar{z} - 2z\bar{z} - z + 2 = 0;$$

(ii)

$$\arg\left[\frac{z-2}{z+i}\right] = \pm\frac{\pi}{2}, \quad (z \neq 2, -i);$$

(iii)

$$\sinh z + 2\cosh z = \frac{\sqrt{3}}{2}.$$

(e) Compute the following limits:

$$\lim_{x\to\infty}\left((3+x^2)^{1/2} - x\right); \quad \lim_{x\to 0}(\cosh\sqrt{x})^{1/x}$$

SECTION B

2. (a) Find the following indefinite integrals:

$$\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\, dx; \quad \int x \log x \; dx; \quad \int \sin^3 x \; dx.$$

(b) Determine a recurrence formula for

$$I_n = \int_0^{\pi/2} \sin^n x \; dx.$$

Hence, or otherwise, find the value of I_{10}.

(c) Find the value of

$$\int_0^{\pi} \frac{d\theta}{5 - 4\cos\theta}.$$

3. (a) Find the Taylor series of the function $\log(1+x)$ about $x = 0$. What is the radius of convergence of this series?

It is required to use the following 3rd-order polynomial approximation to the function $\log(1+x)$ for x-values in the interval $-1/2 \leq x \leq 1/2$, i.e.,

$$\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Using Taylor's theorem, find an estimate of the magnitude of the maximum error incurred in using this polynomial approximation in this interval.

(b) Let

$$f(x) = \frac{e^{2x}}{1-x}.$$

Find an expression for the n-th derivative of $f(x)$, i.e. $\frac{d^n f(x)}{dx^n}$. Hence (or otherwise) show that the coefficient of x^n in the Taylor series expansion of $f(x)$ about $x = 0$ is given by

$$\sum_{j=0}^{n} \frac{2^j}{j!}.$$