UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

FIRST YEAR TEST – JANUARY 2003 0

M1M1 Mathematical Methods (Analytical)

DATE: Monday 6th January 2003 0 TIME: 10.15 am–11.45 am

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers. The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.

Calculators may not be used.

SECTION A

1. (a) Find the derivative of the function $x^2 e^x$ from first principles.

(b) Find the following limits:

$$\lim_{x \to \infty} \left[x \left(\sqrt{3 + x^2} - x \right) \right]; \qquad \lim_{x \to 0} \left[\frac{\sin(\tan 2x)}{\tan x} \right] \tag{1}$$

(c) Find the following indefinite integrals:

$$\int \frac{x^2}{1+x} dx; \qquad \int \frac{\tan^{-1} x}{1+x^2} dx. \tag{2}$$

(d) Find all complex solutions z of the following equations

(i)
$$z^4 + 3z^2 - 4 = 0 \tag{3}$$

(ii)
$$e^{2z} + e^{-z} = 0 \tag{4}$$

(iii)
$$e^{|z|} = -1 \tag{5}$$

(e) Solve the following two differential equations:

$$\frac{dy}{dx} = \frac{x}{y} + \frac{2y}{x}; \quad y(1) = 0. \tag{6}$$

$$\frac{dy}{dx} + \frac{2y}{x^3} = xe^{1/x^2}; \quad y(1) = e. \tag{7}$$

SECTION B

2. Consider the real function $f(x) = e^x \cos x$.

(a) Write $f(x)$ as the sum of an even and an odd function.

(b) Find the first 3 non-zero terms in the Taylor expansion of $f(x)$ about $x = 0$.

(c) By considering the function $\text{Re}[e^{(1+i)x}]$, show that

$$f^{(n)}(0) = 2^{n/2}\cos(n\pi/4), \quad n \geq 0. \tag{8}$$

(d) Use the Leibniz Rule to verify that f satisfies the ordinary differential equation

$$\frac{d^5 f}{dx^5} + 4f = 4e^x \sin x. \tag{9}$$

3. For any integer $n \geq 0$, define the quantity I_n as follows:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx. \tag{10}$$

(a) Show that

$$I_n = \frac{(n-1)}{n} I_{n-2}, \qquad n > 1. \tag{11}$$

Hence show that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2}, \quad n \geq 0. \tag{12}$$

(b) Using the result from part (a), or otherwise, show that when n is even the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^n x \cos^2 x dx \tag{13}$$

is equal to

$$\frac{\pi n!}{2^n[(n/2)!]^2(n+2)}. \tag{14}$$

What is the value of the integral when n is odd?

