Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2007

MC1MF (Test)

Analytical Methods & Analysis

- Affix ONE label to each answer book that you use. DO NOT use the label with your name on it.
- Write your answers in a single answer book, using continuation books if necessary.

- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (i) Let $r_1 = \sqrt{8} - \sqrt{2}$, $r_2 = \sqrt{6}$, $r_3 = 0.10200100020000\ldots$.
Which of these real numbers are irrational?
(a) none of them , (b) r_1 and r_3 only, (c) r_2 and r_3 only , (d)all of them.

(ii) Let $x = 27^{27}$, $y = 81^{19}$ and $z = 9^{99}$. Which of the following is true?
(a) $x < y < z$
(b) $x = z$ and $y < z$
(c) $x < z < y$
(d) $y < x < z$.

(iii) For which integer values of n is $(\sqrt{3} - i)^n$ real?
(a) $n = 0$ only, (b) all integers, (c) multiples of 12 only, (d) multiples of 6 only.

(iv) Which of the following cubics has roots $1 + i$, $1 - i$ and 1?
(a) $4x^3 - 2x^2 + 2x - 1 = 0$
(b) $x^3 - 3x^2 + 4x - 2 = 0$
(c) $x^3 - 2x^2 + 2x - 1 = 0$
(d) $x^3 + 2x^2 - 2x + 4 = 0$.

(v) What is the maximal domain of the real, (single-valued) function

$$f(x) = \sin^{-1}(e^x - 1)?$$

Find the inverse function $f^{-1}(x)$ and express the even part of $f^{-1}(x)$ in as simple a form as possible.

(vi) Sketch the curve

$$y^2 = \frac{4 - x^2}{1 - x^2}.$$

You should identify any stationary points, but need not locate any points of inflection.

(vii) Using any method, evaluate the limits

$$(a) \qquad \lim_{x \to 2} \left(\frac{\sin^2 \pi x}{x^3 - 5x^2 + 8x - 4} \right)$$

$$(b) \qquad \lim_{x \to \infty} \left(\frac{\sin x}{x} + \left(\frac{x+3}{x-1} \right)^x \right)$$

(viii) Evaluate the definite integrals

$$(a) \qquad \int_0^1 \frac{\log(\tan^{-1} x)}{1 + x^2}\, dx \qquad\qquad (b) \qquad \int_0^1 \frac{x + 2}{x^2 + 2x + 2}\, dx$$

SECTION B

2. Show that if $y = \sinh^{-1} x$ then

$$y' = \frac{1}{\sqrt{1+x^2}} \qquad \text{and} \qquad (1+x^2)y'' + xy' = 0$$

Differentiating this equation n times, show that for $n \geqslant 0$

$$y^{(n+2)}(0) = -n^2 y^{(n)}(0)$$

and deduce that the Maclaurin series for y gives

$$\sinh^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(1)^2(3)^2(5)^2 \ldots (2k-1)^2}{(2k+1)!} x^{2k+1}.$$

What is the radius of convergence of this series?

3. (a) Prove that $\sqrt{3}$ is irrational.

(b) Show that $\cos(2\pi/9)$ is a root of the cubic equation $8x^3 - 6x + 1 = 0$. Find the other two roots and deduce that

$$\cos(2\pi/9) + \cos(4\pi/9) + \cos(8\pi/9) = 0.$$

(c) Prove that between any two distinct irrational numbers there is a rational number.

(d) Express the decimal $1.\overline{813}$ as a rational m/n.