M1M1 January Test 2008; Solutions

1(a)

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{dx}{dt}\frac{d}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \qquad [\mathbf{2}]$$

(b) So

$$v\frac{dv}{dx} = v + \frac{1}{x}v^2 \qquad \Longrightarrow \qquad v = 0 \qquad \text{or} \qquad \frac{dv}{dx} = 1 + \frac{1}{x}v.$$

So either $v = 0$ everywhere, or v satisfies a linear equation.

The integrating factor is $\exp(-\int x^{-1}dx) = x^{-1}$. Thus

$$\frac{d}{dx}\left(\frac{v}{x}\right) = \frac{1}{x}$$

Integrating,

$$v = x\log x + cx \qquad (\text{or } v \equiv 0) \qquad [\mathbf{3}](+[\mathbf{1}])$$

(c) If $v = 1$ when $x = 1$ we can't have the $v \equiv 0$ solution and must have $c = 1$. So

$$\frac{dx}{dt} = x(\log x + 1) \qquad \Longrightarrow \qquad t = \int \frac{dx}{x(\log x + 1)}$$

Substituting $u = \log x$ (or spotting that this is a logarithmic derivative) we find

$$t = \log(\log x + 1) + d = \log(\log x + 1)$$

imposing $x = 0$ at $t = 0$. Thus, as required,

$$x = \exp(e^t - 1) \equiv f(t). \qquad [\mathbf{3}]$$

(d) As t takes all values, e^t takes all positive values. Thus x takes all values with $x > e^{-1}$ The inverse function is $t = \log(1+\log x) = f^{-1}(x)$. This is defined provided both logs have positive arguments, which requires $\log x > -1$ or $x > e^{-1}$. **[2]**

(e) The curve $x = f(t)$ has stationary points where $f'(t) = 0$, and inflection points where $f''(t) = 0$. Now $f'(t) = e^t \exp(e^t - 1)$. This is never zero as real exponentials never are. Similarly $f''(t) = (e^t + e^{2t})\exp(e^t - 1) > 0$ always. The curve has no stationary points or inflection points. **[2]**

(f) We know $x(0) = 1$, $x'(0) = 1$ and from the original equation $x''(0) = x'(0) + [x'(0)]^2/x(0) = 2$ So the Maclaurin series is

$$x(t) = 1 + t + \tfrac{1}{2}(2)t^2 + O(t^3) = 1 + t + t^2 + O(t^3). \qquad [\mathbf{2}]$$

(g) As $t \to 0$, we know $x \to 1$. Thus (or by other methods)

$$\lim_{t\to 0}\left[\frac{\log f(t)}{(f(t)-1)^{2/3}}\right] = \lim_{x\to 1}\left[\frac{\log x}{(x-1)^{2/3}}\right] = \lim_{x\to 1}\left[\frac{(x-1) + O((x-1)^2)}{(x-1)^{2/3}}\right] = 0. \qquad [\mathbf{2}]$$

(h) When $t = 2i$, we have $x = \exp(e^{2i} - 1)$. Now

$$\exp(e^{2i} - 1) = \exp((\cos 2 - 1 + i \sin 2)) = \exp(\cos 2 - 1)(\cos(\sin 2) + i \sin(\sin 2))$$

So the real part is

$$\Re e(x) = e^{(\cos 2 - 1)}(\cos(\sin 2)). \qquad \textbf{[3]}$$

Total : **[20]**

2. **(a)** The Mean Value Theorem states that if $f(x)$ is continuous in an interval $[a, b]$ and differentiable in (a, b) then there exists a ξ in $a < \xi < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi). \qquad \textbf{[2]}$$

So if $f(a) = 0 = g(a)$ then $f(x) = (x-a)f'(\xi)$ and $g(x) = (x-a)g'(\eta)$ for some ξ and η with $a < \xi < x$ and $a < \eta < x$. (Note ξ and η are different, in general.) **[2]**

Now as $x \to a$ it is clear that $\xi \to a$ and also $\eta \to a$. Thus

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to a} \left[\frac{f'(\xi)}{g'(\eta)} \right] = \frac{f'(a)}{g'(a)}, \qquad \textbf{[2]}$$

since the derivatives are continuous and the denominator non-zero.

(b)(i) Both numerator and denominator are zero. Assuming both limits exist,

$$\lim_{x \to 1/2} \left[\frac{\log(\sin \pi x)}{(2x - 1)^2} \right] = \lim_{x \to 1/2} \left[\frac{\pi \cos \pi x / \sin(\pi x)}{4(2x - 1)} \right] = \tfrac{1}{4}\pi \lim_{x \to 1/2} \left[\frac{\cos \pi x}{2x - 1} \right]. \qquad \textbf{[3]}$$

Once more this is of form "0/0". Using de l'Hôpital's rule again, assuming the limits exist

$$\lim_{x \to 1/2} \left[\frac{\cos \pi x}{2x - 1} \right] = \lim_{x \to 1/2} \left[\frac{-\pi \sin \pi x}{2} \right] = -\tfrac{1}{2}\pi. \qquad \textbf{[2]}$$

Since this latter limit exists, so does the intermediate one, and hence so does the original limit. We deduce

$$\lim_{x \to 1/2} \left[\frac{\log(\sin \pi x)}{(2x - 1)^2} \right] = -\frac{\pi^2}{8}. \qquad \textbf{[2]}$$

(ii) Once more $f(0) = 0 = g(0)$. Now $g'(x) = \sin x$ and $f'(x) = 3x^2 \sin(1/x) - x \cos(1/x)$. So $f'(0) = 0 = g'(0)$. However

$$\lim_{x \to 0} \left[\frac{f'(x)}{g'(x)} \right] = \lim_{x \to 0} \left[3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right] \left[\frac{x}{\sin x} \right]$$

and this does not tend to a limit, because of the $\cos(1/x)$ term. We cannot use de l'Hôpital's rule here. **[4]**

However, going back to the original limit, since $|\sin(1/x)| \leqslant 1$, it is clear that $f(x) = O(x^3)$ as $x \to 0$. As $g(x) = 1 - \cos(x) = \frac{1}{2}x^2 + O(x^4)$, it follows that

$$\lim_{x\to 0}\left[\frac{f(x)}{g(x)}\right] = 0. \qquad [\mathbf{3}]$$

Total : [**20**]

3.(a) If $f(x) = \log(x+a)$, then $f'(x) = (x+a)^{-1}$, $f''(x) = -(x+a)^{-2}$ and

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(x+a)^n} \qquad \text{for } n > 1. \qquad [\mathbf{2}]$$

Thus

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!} = f(0) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x}{a}\right)^n = \log a + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x}{a}\right)^n. \qquad [\mathbf{2}]$$

The Radius of Convergence follows from the ratio test: we need

$$\left|\frac{x}{a}\right|\lim_{n\to\infty}\left[\frac{(n+1)}{n}\right] < 1 \quad \Longrightarrow |x| < a,$$

so radius of convergence is a. [**2**]

(b) If the series holds when $a = i$, then

$$\log(x+i) = \log i - \sum_{n=1}^{\infty}\frac{(ix)^n}{n}.$$

Now $\log(re^{i\theta}) = \log r + i\theta(+2k\pi i)$. So $\log i = \frac{1}{2}\pi i$. Writing $x + i = r(\cos\theta + i\sin\theta)$ we have $x = r\cos\theta$ and $1 = r\sin\theta$ so that $r = \sqrt{1+x^2}$ and $\sin\theta = 1/r$ (with $\cos\theta > 0$). Taking the imaginary part we have

$$\theta + 2k\pi = \tfrac{1}{2}\pi - \left[x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots\right] = \tfrac{1}{2}\pi + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2m-1}.$$

As we want $0 < \theta < \frac{1}{2}\pi$, choose $k = 0$. Combining things, we have

$$\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \tfrac{1}{2}\pi - x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 + \ldots \qquad [\mathbf{8}]$$

(c) We know that $\frac{d}{du}(\sin^{-1}u) = (1-u^2)^{-1/2}$. So

$$g'(x) = \frac{1}{(1-(1/(1+x^2))^{1/2}}\frac{(-1/2)2x}{(1+x^2)^{3/2}} = \frac{-x}{(x^2)^{1/2}(1+x^2)} = -\frac{1}{1+x^2} \qquad [\mathbf{4}]$$

assuming $x > 0$. Now

$$-\frac{1}{1+x^2} = -1 + x^2 - x^4 + \ldots = \frac{d}{dx}\left[\tfrac{1}{2}\pi - x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 + \ldots\right] \qquad [\mathbf{2}]$$

Total : [**20**]