M1M1 January test 2007: SOLUTIONS

1. (a) Since $\sin^{-1} x$ is defined for $|x| \leqslant 1$ we see $f(x)$ is defined for $-1 \leqslant e^x - 1 \leqslant 1$ or $0 \leqslant e^x \leqslant 2$. Thus the maximal domain of f is $\log 2 \geqslant x > -\infty$. **[1]**

Writing

$$y = \sin^{-1}(e^x - 1) \qquad \Longrightarrow \qquad e^x = 1 + \sin y$$

or

$$x = \log(1 + \sin y) \equiv f^{-1}(y) \qquad [\mathbf{1}]$$

Thus if $h(x)$ is the even part of $f^{-1}(x)$, then $h(x) = \frac{1}{2}[f^{-1}(x) + f^{-1}(-x)]$ or

$$h(x) = \tfrac{1}{2}[\log(1 + \sin x) + \log(1 - \sin x)] = \tfrac{1}{2}\log(1 - \sin^2 x) = \log|\cos x|. \qquad [\mathbf{1}]$$

(b) We say $f(x)$ is differentiable at $x = a$ if the limit (with ε not necessarily positive)

$$\lim_{\varepsilon \to 0} \left[\frac{f(a + \varepsilon) - f(a)}{\varepsilon}\right] \qquad \text{exists.} \qquad [\mathbf{1}]$$

Now if $f(x) = 1/(1 + x)$, then

$$\begin{aligned} f'(x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\frac{1}{1 + x + \varepsilon} - \frac{1}{1 + x}\right) = \lim_{\varepsilon \to 0}\left[\frac{1 + x - (1 + x + \varepsilon)}{\varepsilon(1 + x)(1 + x + \varepsilon)}\right] \\ &= \lim_{\varepsilon \to 0}\left[\frac{-1}{(1 + x)(1 + x + \varepsilon)}\right] = -\frac{1}{(1 + x)^2} \end{aligned}. \qquad [\mathbf{2}]$$

(c) We have

$$y^2 = \frac{(4 - x^2)}{(1 - x^2)} = 1 + \frac{3}{1 - x^2}.$$

As only x^2 and y^2 appear in the equation, we know the curve will be symmetric in both the x-axis and y-axis. [Furthermore as $(x^2 - 1)(y^2 - 1) = -3$ is a symmetrical expression in x^2 and y^2 the curve will also be symmetrical in the lines $y = \pm x$.] Now $(4 - x^2)/(1 - x^2)$ is positive only for $x^2 > 4$ or $x^2 < 1$ and so the curve only exists in these regions. y is zero at $x = \pm 2$ and infinite at $x = \pm 1$. As $|x| \to \infty$ we see $y^2 \to 1$. Differentiating, we have

$$2yy' = \frac{6x}{(1 - x^2)^2} \qquad = 0 \quad \text{at } x = 0 \text{ when } y = \pm 2$$

As x increases through zero, y' changes from $+$ to $-$ if $y < 0$, so there is a maximum at $(0, -2)$ and a minimum at $(0, +2)$. Putting all this together, we get the highly symmetric curve: (See last page **[3]**) (d)(i) Substituting in $x = 2$ we see the expression is of the form "0/0". So using de l'Hôpital's rule

$$\lim_{x\to 2}\left(\frac{\sin^2 \pi x}{x^3 - 5x^2 + 8x - 4}\right) = \lim_{x\to 2}\left(\frac{2\pi \sin \pi x \cos \pi x}{3x^2 - 10x + 8}\right) = \lim_{x\to 2}\left(\frac{2\pi \sin \pi x}{3x^2 - 10x + 8}\right).$$

This is still of the form "0/0", and so using de l'Hôpital once more

$$\lim_{x\to 2}\left(\frac{2\pi \sin \pi x}{3x^2 - 10x + 8}\right) = \lim_{x\to 2}\left(\frac{2\pi^2 \cos \pi x}{6x - 10}\right) = \frac{2\pi^2}{2} = \pi^2 \qquad [\mathbf{2}]$$

(ii) taking each part of the expression separately, we see $|\sin(x)/x| \leqslant 1/x \to 0$ as $x \to \infty$, and so the first part has limit zero. Then defining

$$u = \left(\frac{x+3}{x-1}\right)^x \implies \log u = x \log \frac{x+3}{x-1} = x \log\left[1 + \frac{4}{x-1}\right] = x\left[\frac{4}{x-1} + O\left(\frac{4}{x-1}\right)^2\right]$$

Thus as $x \to \infty$, we see $\log u \to 4$ and so $u \to e^4$. The limit is thus $0 + e^4 = e^4$. **[2]**

(e)(i) Writing $-1+i$ in the form $re^{i\theta}$, we have $r^2 = 2$ and $\cos\theta = -1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$. Thus $\theta = 3\pi/4 + 2k\pi$ where k is any integer and so:

$$e^z = \sqrt{2}\, e^{i(3\pi/4 + 2k\pi)}$$

It follows that

$$z = \tfrac{1}{2}\log 2 + i(3\pi/4 + 2k\pi).$$

(ii) If $z = x + iy$ with x, y real, then the given equation is $2xy = x^2 + y^2$ or $(x-y)^2 = 0$. Thus $x = y = c$, say. The general solution is therefore $z = c(1+i)$. ([**3**], 2 for either part (i) or part (ii))

(f)(i) Since we know that $1/(1+x^2)$ is the derivative of $\tan^{-1} x$, we make the substitution $u = \tan^{-1} x$, noting that when $x = 1$, $u = \pi/4$

$$I = \int_0^1 \frac{\log(\tan^{-1} x)}{1+x^2}\, dx = \int_0^{\pi/4} \log u \, du.$$

Then integrating by parts, treating $\log u$ as the product of 1 and $\log u$, we have

$$I = \int_0^{\pi/4} \log u\, du = \Big[u \log u\Big]_0^{\pi/4} - \int_0^{\pi/4} u\left(\frac{1}{u}\right) du = \tfrac{1}{4}\pi\left(\log \tfrac{1}{4}\pi - 1\right). \qquad [\mathbf{2}]$$

(ii) Completing the square in the denominator and then substituting $(x+1) = \tan\theta$,

$$\int_0^1 \frac{x+2}{x^2+2x+2}\, dx = \int_0^1 \frac{\frac{1}{2}(2x+2)+1}{x^2+2x+2}\, dx = \tfrac{1}{2}\Big[\log(x^2+2x+2)\Big]_0^1 + \int_0^1 \frac{dx}{(x+1)^2+1}$$

$$= \tfrac{1}{2}(\log 5 - \log 2) + \Big[\theta\Big]_{\tan^{-1} 1}^{\tan^{-1} 2} = \tfrac{1}{2}\log(5/2) + \tan^{-1} 2 - \tfrac{1}{4}\pi.$$

[2]

2. If $y = \sinh^{-1} x$ so $x = \sinh y$ and $dx/dy = \cosh y = \sqrt{1+x^2}$. Therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad \Longrightarrow \quad \frac{d^2y}{dx^2} = \frac{(-1/2)(2x)}{(1+x^2)^{3/2}} = -\frac{x}{1+x^2}\frac{dy}{dx} \qquad [\mathbf{5}]$$

Regrouping,

$$(1+x^2)y'' + xy' = 0$$

Differentiating n times using Leibniz's formula,

$$(1+x^2)y^{(n+2)} + n(2x)y^{(n+1)} + \frac{n(n-1)}{2}(2)y^{(n)} + xy^{(n+1)} + ny^{(n)} = 0$$

Evaluating this equation at $x = 0$, we have

$$y^{(n+2)}(0) + (n(n-1)+n)y^{(n)}(0) = 0 \quad \Longrightarrow \quad y^{(n+2)}(0) + n^2 y^{(n)}(0) = 0 \qquad [\mathbf{7}]$$

as required. Now $y(0) = 0$ and from the above formula $y'(0) = 1$. It follows that all even derivatives are zero and $y^{(2k+1)}(0) = -(2k-1)^2 y^{(2k-1)}(0)$. So using the Maclaurin series,

$$\sinh^{-1}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{(1)^2(3)^2(5)^2\dots(2k-1)^2}{2k+1!}x^{2k+1} = x - \frac{x^3}{6} + \frac{9x^5}{5!} + \dots \qquad [\mathbf{4}]$$

Using the ratio test, the series converges if

$$1 > \lim_{k\to\infty}\left|\frac{y^{(2k+1)}(0)x^{2k+1}/(2k+1)!}{y^{(2k-1)}(0)x^{2k-1}/(2k-1)!}\right| = \lim_{k\to\infty}\left|\frac{-(2k-1)^2x^2}{(2k+1)(2k)}\right| = x^2$$

Therefore the series has radius of convergence 1. **[4]**

3. As the ODE is linear, we use the integrating factor

$$I = \exp\left[\int \frac{4\sin x\, dx}{5+4\cos x}\right] = \exp[-\log(5+4\cos x)] = (5+4\cos x)^{-1} \qquad [\mathbf{5}]$$

Multiplying the ODE by I, we obtain

$$\frac{d}{dx}\left(\frac{y}{5+4\cos x}\right) = \frac{3}{2}\int \frac{dx}{5+4\cos x} \qquad [\mathbf{3}]$$

Now using the substitution $t = \tan \frac{1}{2}x$, we have $\cos x = (1-t^2)/(1+t^2)$ and $dx/dt = 2/(1+t^2)$ so

$$\int \frac{dx}{5+4\cos x} = \int \frac{2dt}{5(1+t^2)+4(1-t^2)} = \int \frac{2dt}{9+t^2} = \frac{2}{3}\tan^{-1}(t/3) + c, \qquad [\mathbf{7}]$$

where c is an arbitrary constant. Therefore integrating the ODE, we have

$$\frac{y}{5+4\cos x} = \tan^{-1}[\tfrac{1}{3}\tan(x/2)] + c \quad \Longrightarrow \quad y = (5+4\cos x)\left(\tan^{-1}[\tfrac{1}{3}\tan(x/2)] + c\right) \quad [\mathbf{2}]$$

If $y(0) = 0$, then $0 = 9(0+c)$ or $c = 0$, giving the required solution. **[3]**