M1M1: January Test 2005: SOLUTIONS

A1.(a) First, rewrite the function as

$$f(x) = \frac{x^4 - 1 + 2}{x^4 - 1} = 1 + \frac{2}{x^4 - 1}.$$

Note that $f(x)$ is even. It never vanishes. It has vertical asymptotes at $x = \pm 1$. $y \to 1$ as $x \to \pm\infty$ so $y = 1$ is a horizontal asymptote. The derivative is

$$\frac{df}{dx} = -\frac{8x^3}{x^4 - 1}$$

so there is a stationary point at $x = 0$. See Figure 1.

(b) The function is

$$\frac{e^{4x} + 1}{e^{4x} - 1} = \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}} = \mathrm{cotanh} 2x$$

Its graph has a vertical asymptote at $x = 0$ while it tends to 1 as $x \to \infty$ and -1 as $x \to -\infty$. The function is odd. It is easy to verify that it has no stationary points. See Figure 2.

A2. (a) On use of the binomial expansion

$$e^{(1+x^3)^{1/2}} = e^{1+\frac{x^3}{2}-\frac{x^6}{8}+\frac{x^9}{16}\cdots}$$

Define

$$X = \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} + \ldots$$

Then

$$\begin{aligned} e^{1+X} &= e\left(1 + X + \frac{X^2}{2!} + \ldots\right) \\ &= e\left(1 + \left(\frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} + \ldots\right) + \frac{1}{2!}\left(\frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} + \ldots\right)^2 + \right. \\ &\qquad \left. + \frac{1}{3!}\left(\frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} + \ldots\right)^3\right) \\ &= e + \frac{ex^3}{2} + \frac{ex^9}{48} \end{aligned}$$

(b) Note that

$$\log\left(1 + \log(1+x)\right) = \log\left(1 + x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right).$$

Let

$$X = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Then

$$\begin{aligned}
\log\Big(1 + \log(1+x)\Big) &= \log\Big(1 + X\Big) \\
&= X - \frac{X^2}{2} + \frac{X^3}{3} + \dots \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right) - \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right)^2 \\
&\qquad + \frac{1}{3}\left(x - \frac{x^2}{2} + \dots\right)^3 + \dots \\
&= x - x^2 + \frac{7x^3}{6} + \dots
\end{aligned}$$

A3. (a) On use of the substitution $u = e^x$, integral becomes

$$\int_1^e \frac{du}{u^2+1} = \Big[\tan^{-1}(u)\Big]_1^e = \tan^{-1} e - \frac{\pi}{4}.$$

(b) Note that

$$\int \frac{\cosh 2x}{\sinh x} dx = \int \frac{e^{2x} + e^{-2x}}{e^x - e^{-x}} dx.$$

On use of the substitution $u = e^x$, integral becomes

$$\int \frac{u^4+1}{u^2(u^2-1)} du = \int 1 + \frac{u^2+1}{u^2(u^2-1)} du$$

Putting the integrand into partial fraction form yields

$$\int 1 - \frac{1}{u^2} + \frac{1}{u-1} - \frac{1}{u+1} du.$$

Integral is therefore

$$u + \frac{1}{u} + \log|u-1| - \log|u+1| + c = e^x + e^{-x} + \log\left|\frac{e^x-1}{e^x+1}\right| + c,$$

or

$$2\cosh x + \log|\tanh(x/2)| + c.$$

A4. Note that

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = 3.$$

Rearrangement yields

$$e^{2z} = -2 = e^{\log 2 + i\pi + 2k\pi i}$$

where k is any integer. We therefore identify

$$z = \frac{1}{2}\log 2 + \frac{i\pi}{2} + k\pi i$$

where k is any integer.

(b) Squaring the equation yields

$$|z-i|^2 + |z+i|^2 + 2|z-i||z+i| = 8$$

Expanding

$$x^2 + (y-1)^2 + x^2 + (y+1)^2 + 2\sqrt{(x^2-y^2+1)^2 + 4x^2y^2} = 8$$

Further algebra leads to

$$x^2 + \frac{y^2}{2} = 1$$

which is an ellipse centred at the origin with semi-major axis of length $\sqrt{2}$ along the imaginary axis and semi-minor axis of unit length along the x-axis.

Note: any correct geometrical argument is also acceptable (although this was not taught in lectures so it is unlikely they will know any).

A5. The ordinary differential is linear with integrating factor given by

$$e^{\int \frac{1}{x^2}dx} = e^{-x^{-1}}$$

Therefore, equation becomes

$$\frac{d(ye^{-x^{-1}})}{dx} = 1.$$

Integration yields

$$e^{-x^{-1}}y = x + c$$

where $c = -1$ by the condition that $y(1) = 0$. Solution is

$$y(x) = (x-1)e^{1/x}.$$

B1.(a) Need to compute

$$\lim_{\epsilon\to 0}\left(\frac{\log(1+\sqrt{x+\epsilon}) - \log(1+\sqrt{x})}{\epsilon}\right).$$

But

$$\begin{aligned}
\log\left(1+\sqrt{x+\epsilon}\right) &= \log\left(1+\sqrt{x}\left(1+\frac{\epsilon}{x}\right)^{1/2}\right) \\
&= \log\left(1+\sqrt{x}+\frac{\epsilon}{2\sqrt{x}}+...\right) \\
&= \log\left((1+\sqrt{x})\left(1+\frac{\epsilon}{2\sqrt{x}(1+\sqrt{x})}+....\right)\right) \\
&= \log(1+\sqrt{x}) + \log\left(1+\frac{\epsilon}{2\sqrt{x}(1+\sqrt{x})}+...\right) \\
&= \log(1+\sqrt{x}) + \frac{\epsilon}{2\sqrt{x}(1+\sqrt{x})}+...
\end{aligned}$$

Therefore, required limit is

$$\frac{1}{2\sqrt{x}(1+\sqrt{x})}.$$

(b) Let x denote the semi-width of the rectangle. The area A is given by $A = 4xy$ where y is the semi-height. We require to maximize A, or equivalently A^2, with respect to x. But

$$A^2 = 16x^2y^2 = 16x^2\left(9 - \frac{9x^2}{4}\right).$$

We therefore want to maximize the quantity

$$A' = x^2\left(1 - \frac{x^2}{4}\right)$$

with respect to x. Taking derivatives

$$\frac{dA'}{dx} = 2x\left(1 - \frac{x^2}{4}\right) - \frac{x^3}{2} = x(2 - x^2)$$

which vanishes when $x = 0$ and $x = \sqrt{2}$. First is a minimum, second is the required maximum. Maximal area A is therefore $A = 12$.

(c) Let

$$f(x) = \frac{e^x}{1 - x}$$

On use of the Leibniz rule,

$$f^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{dx^j}\left(\frac{1}{1-x}\right) \frac{d^{n-j}}{dx^{n-j}} e^x.$$

But it is easy to show that

$$\frac{d^j}{dx^j}\left(\frac{1}{1-x}\right) = \frac{j!}{(1-x)^{j+1}}.$$

Thus

$$f^n(x) = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{e^x}{(1-x)^{j+1}}.$$

On use of the Taylor series formula, the coefficient a_n of x^n is

$$a_n = \frac{f^n(0)}{n!} = \sum_{j=0}^{n} \frac{1}{(n-j)!}.$$

B2. (a) Putting the integrand in partial fraction form yields

$$\int \frac{x^3+1}{x^3-1} dx = \int 1 + \frac{2}{x^3-1} dx = \int 1 + \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} dx$$

Straightforward algebra leads to

$$\int 1 + \frac{2}{3}\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right) dx$$
$$= \int 1 + \frac{2}{3}\left(\frac{1}{x-1}\right) - \frac{2}{3}\left(\frac{x+1/2}{x^2+x+1}\right) - \left(\frac{1}{x^2+x+1}\right) dx$$

Note that the third contribution to the integrand is a logarithmic derivative. Concerning the fourth contribution,

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2+3/4}$$

and the substitution $(\sqrt{3}/2)u = x + 1/2$ leads to this integral evaluating to

$$\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right).$$

Final integral becomes

$$x + \frac{2}{3}\log|x-1| - \frac{1}{3}\log|x^2+x+1| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + c.$$

(b) On use of a trigonometric identity $\cot^2 x + 1 = \operatorname{cosec}^2 x$,

$$\begin{aligned} I_n = \int_{\pi/4}^{\pi/2} \cot^n x dx &= \int_{\pi/4}^{\pi/2} \cot^{n-2} x \left(\operatorname{cosec}^2 x - 1\right) dx \\ &= -\left[\frac{\cot^{n-1} x}{n-1}\right]_{\pi/4}^{\pi/2} - I_{n-2} \\ &= \frac{1}{n-1} - I_{n-2}. \end{aligned}$$

(c) Let

$$u = \frac{dy}{dx}.$$

Then

$$\frac{du}{dx} + u = x.$$

Integrating factor is e^x. Therefore equation becomes

$$\frac{d}{dx}\left(e^x u\right) = xe^x.$$

Integration by parts on right hand side leads to

$$e^x u = (x-1)e^x + c.$$

Condition $y'(0) = 0$ leads to $c = 1$. Then

$$\frac{dy}{dx} = x - 1 + e^{-x}.$$

A further integration yields

$$y = \frac{x^2}{2} - x - e^{-x} + d$$

where condition $y(0) = 0$ leads to $d = 1$. Final solution is

$$y = \frac{x^2}{2} - x - e^{-x} + 1.$$

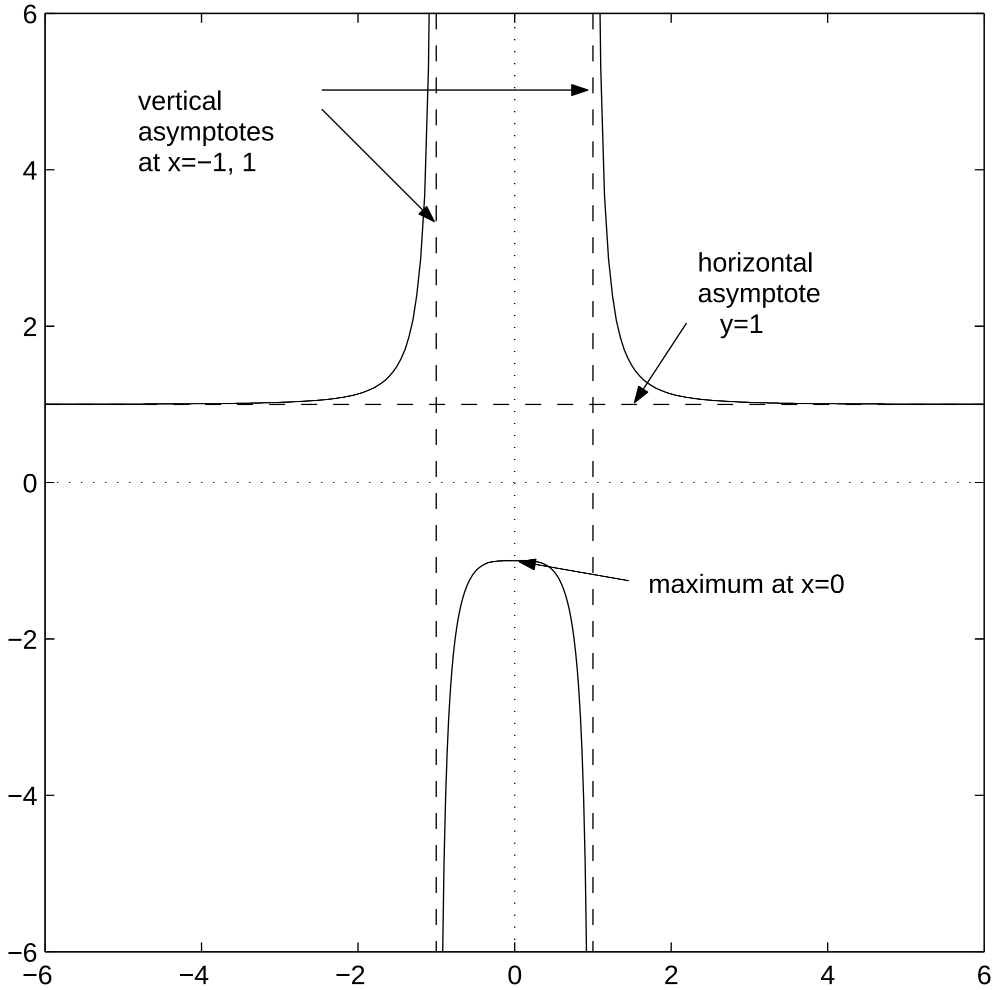

Figure 1: Graph for question A1(a)

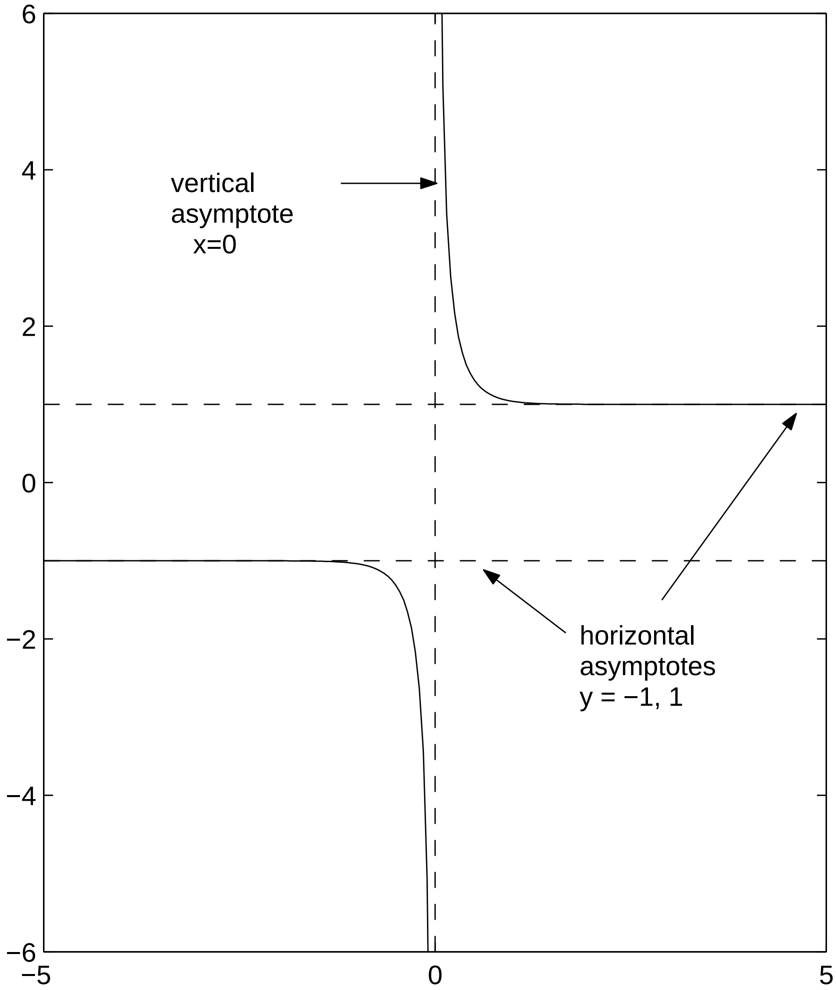

Figure 2: Graph for question A1(b)