Question 2. In lectures we showed the curvature, K, of a curve in the (x, y)-plane was

$$K = \frac{y''}{(1 + y'^2)^{3/2}}.$$

(a) If the curve is given parametrically in the form $x = f(t)$, $y = g(t)$, obtain a formula for K in terms of f, g and t.

(b) In suitable units, Prof Corti's nose is described in polar coordinates by $r = \cos\theta$. By defining x and y parametrically in terms of θ and using part (a), find its curvature K, simplifying your answer as much as possible.

(c) Sketch the curve $r = \cos\theta$.

Answer. (a) We have $y'(x) = g'(t)/f'(t)$. Differentiating w.r.t. x, we have

$$y'' = \frac{dt}{dx}\left[\frac{f'g'' - f''g'}{f'^2}\right] = \left[\frac{f'g'' - f''g'}{f'^3}\right].$$

Thus

$$K = \left[\frac{f'g'' - f''g'}{f'^3}\right]\frac{1}{(1 + (g'/f')^2)^{3/2}} = \frac{f'g'' - f''g'}{(f'^2 + g'^2)^{3/2}}. \qquad \textbf{(3 marks)}$$

(b) We have $x = r\cos\theta$. $y = r\sin\theta$ so that we can define

$$f(\theta) = \cos^2\theta, \qquad g(\theta) = \cos\theta\sin\theta.$$

Then writing $c = \cos\theta$ and $s = \sin\theta$, we have $f' = -2sc$, $g' = (c^2 - s^2)$, $f'' = 2(s^2 - c^2)$, $g'' = -4sc$. Substituting, we have

$$K = \frac{(-2sc)(-4sc) - 2(s^2 - c^2)(c^2 - s^2)}{[4s^2c^2 + (c^2 - s^2)^2]^{3/2}} = \frac{2s^4 + 2c^4 + 4s^2c^2}{(c^2 + s^2)^3} = 2. \qquad \textbf{(5 marks)}$$

(c) Constant curvature means a circle. Symmetry about $\theta = 0$, and passing through $(0, 1)$ when $\theta = 0$ and $(0, 0)$ when $\theta = \frac{1}{2}\pi$ means a circle of radius $\frac{1}{2}$, centre $(\frac{1}{2}, 0)$. **(2 marks)**

Total : 10

[Notes for markers: In the first part, some will miss a factor of f'. There will be some algebraic errors - use your judgement how much credit to give. If all markers agree, you can redistribute marks between the parts. Indeed, you may to a large extent do as you choose, but of course you must be consistent across all the scripts. Remember they only have 15 minutes for this question. The students will eventually see a copy of this sheet. Don't forget to initial your work to help in the "Meet your Marker" sessions.]