Question 1. Use the Mean Value Theorem for the function $\log(1+x)$ over $[0,t]$ to show that for $0<t$,

$$t > \log(1+t) > \frac{t}{1+t}.$$

Obtain a similar result for the interval $[-t,0]$ assuming $0<t<1$ and deduce that

$$\frac{2t-t^2}{1-t} > \log\left[\frac{1+t}{1-t}\right] > \frac{2t+t^2}{1+t}.$$

Answer. The MVT for $\log(1+x)$ between $x=0$ and $x=t$, states that there exists a ξ, for $0<\xi<t$ such that

$$\frac{\log(1+t)-\log(1)}{t-0} = \frac{1}{1+\xi}. \qquad \textbf{(2 marks)}$$

But as $1/(1+x)$ is a decreasing function,

$$1 > \frac{1}{1+\xi} > \frac{1}{1+t} \quad \Longrightarrow \quad 1 > \frac{\log(1+t)}{t} > \frac{1}{1+t}.$$

As $t>0$ we can multiply the inequality by it, to obtain

$$t > \log(1+t) > \frac{t}{1+t}. \qquad \textbf{(2 marks)}$$

Using the same function over the interval $[-t,\,0]$, there is an η in $-t<\eta<0$ such that

$$\frac{\log(1)-\log(1-t)}{0-(-t)} = \frac{1}{1+\eta} \qquad \text{and} \qquad \frac{1}{1-t} > \frac{1}{1+\eta} > 1.$$

Thus for $1>t>0$

$$\frac{t}{1-t} > -\log(1-t) > t. \qquad \textbf{(4 marks)}$$

Adding the two inequalities, in the range where both apply $(0<t<1)$

$$\frac{2t-t^2}{1-t} = t + \frac{t}{1-t} > \log\left[\frac{1+t}{1-t}\right] > t + \frac{t}{1+t} = \frac{2t+t^2}{1+t}. \qquad \textbf{(2 marks)}$$

Total : 10

[Notes for markers: Clarity of argument is important here. If they use the same symbol for ξ, η in the two parts warn them, but do not deduct a mark unless your impression is that they think the values are the same. If all markers agree, you can redistribute marks between the parts. Indeed, you may to a large extent do as you choose, but of course you must be consistent across all the scripts. Remember they only have 15 minutes for this question. The students will eventually see a copy of this sheet. Don't forget to initial your work to help in the "Meet your Marker" sessions.]