Question 3. After a Monday test, a maths lecturer (of your choice) mysteriously falls from the top of the Queen's Tower. After a time $t > 0$ he/she has fallen a distance x, where

$$x = \frac{1}{k} \log \left[\cosh \left(t\sqrt{k} \right) \right],$$

and k is a positive constant related to air resistance (we have assumed gravity $g = 1$.)
(a) Find the first two terms in an approximation for x if k is small.
(b) If k is large (perhaps the lecturer had a parachute), show that

$$x \simeq U(t - t_0)$$

where U and t_0 are constants you should find in terms of k.

Answer. (a) We have $\cosh \xi = \frac{1}{2}(e^{\xi} + e^{-\xi}) = 1 + \frac{1}{2}\xi^2 + 1/(24)\xi^4 + O(\xi^6)$ **(1 mark)**
and so

$$x = \frac{1}{k} \log \left[1 + \tfrac{1}{2}kt^2 + \tfrac{1}{24}k^2t^4 + O(k^3t^6) \right] = \frac{1}{k} \log(1 + u), \qquad \text{say.}$$

Now

$$\log(1 + u) = u - \tfrac{1}{2}u^2 + O(u^3) \qquad \textbf{(1 mark)}$$

so

$$x = \frac{1}{k} \left[\left(\tfrac{1}{2}kt^2 + \tfrac{1}{24}k^2t^4 \right) - \tfrac{1}{2} \left(\tfrac{1}{2}kt^2 + \tfrac{1}{24}k^2t^4 \right)^2 + \ldots \right] = \tfrac{1}{2}t^2 - \tfrac{1}{12}kt^4 + \ldots \qquad \textbf{(4 marks)}$$

(b) When ξ is large, $\cosh \xi = \frac{1}{2}e^{\xi}+$ something exponentially small. It follows that $\log \cosh \xi \simeq \log(\frac{1}{2}) + \log(e^{\xi}) = -\log 2 + \xi$. So when k is large,

$$x \simeq \frac{1}{k}(-\log 2 + t\sqrt{k}) \quad \Longrightarrow \quad U_0 = \frac{1}{\sqrt{k}} \quad \text{and} \quad t_0 = \frac{\log 2}{\sqrt{k}} \qquad \textbf{(4 marks)}$$

Total 10

[**Note:** Strictly speaking, rather than k small or large we should consider kt^2 being small or large. Those of you who did A-level mechanics should have expected the $\frac{1}{2}(g)t^2$ term in part (a). In part (b), U_0 is called the *terminal velocity,* as it is the limit of your speed as $t \to \infty$. For a human falling through air this is about 100-200 mph, which usually is "terminal". For smaller creatures such as mice, k is larger so that U_0 is much smaller. Small animals may survive a fall of any height.]

[Notes for markers: Give marks both for method as well as numerical exactness. I expect many will forget the u^2 term in the log expansion You may award partial credit for good attempts and redistribute marks between the parts as you choose, provided all markers agree. Indeed, to a large extent, you may do as you choose, but of course you must be consistent across all the scripts. The students will eventually see a copy of this sheet. Don't forget to initial your work to help in the "Meet your Marker" sessions.]