Question 2. Continuing the results from lectures (which you may assume), suppose now the polynomial $Q(x)$ has a repeated root, r_1 but all the other roots are different, so that $Q = \lambda(x-r_1)^2(x-r_3)(x-r_4)\dots(x-r_N)$. Assuming the partial fraction expansion

$$\frac{P(x)}{Q(x)} = R(x) + \frac{A_1}{(x-r_1)^2} + \frac{B}{x-r_1} + \frac{A_3}{x-r_3} + \dots \frac{A_N}{x-r_N},$$

where P and R are polynomials, find A_1 in terms of r_1, P and the second derivative of Q.

Given that in general

$$B = \frac{d}{dx}\left[\frac{P(x)(x-r_1)^2}{Q(x)}\right]_{x=r_1},$$

find B when $P(x) = ax+b$ and $Q(x) = (x-r_1)^2(x-r_3)$.

Deduce that $B + A_3$ does not depend on r_1, r_3, a or b.

Answer. Multiplying up by $Q(x)$, and setting $x = r_1$we have

$$P(r_1) = 0 + A_1\lambda(r_1-r_3)(r_1-r_4)\dots(r_1-r_N) + 0 + 0\dots \qquad \textbf{(2 marks)}$$

Now

$$Q''(r_1) = 2\lambda(r_1-r_3)(r_1-r_4)\dots(r_1-r_N)$$

It follows that

$$A_1 = \frac{2P(r_1)}{Q''(r_1)}. \qquad \textbf{(2 marks)}$$

Substituing in the given formula, we have

$$B = \frac{d}{dx}\left[\frac{ax+b}{x-r_3}\right]_{x=r_1} = \frac{d}{dx}\left[\frac{a(x-r_3)+b+ar_3}{x-r_3}\right]_{x=r_1} = -\frac{b+ar_3}{(r_1-r_3)^2}, \qquad \textbf{(3 marks)}$$

Now from lectures (or by multiplying by Q_1 and setting $x = r_3$),

$$A_3 = \frac{P(r_3)}{Q'(r_3)} = \frac{ar_3+b}{(r_3-r_1)^2} \quad \Longrightarrow \quad B + A_3 = 0. \qquad \textbf{(3 marks)}$$

This is clearly independent of everything.

Total 10

[Notes for markers: This question derives somewhat from Thursday's lecture, but some may get confused by the notation. Some may find the partial fraction expansion using their own methods. That is acceptable, but be less sympathetic to algebraic slips if they "do it their way." Anyone who explains that **if** *$B+A_3$ is a constant it has to be zero (else it would double if a and b double) deserves a mark for intelligence. You may award partial credit for good attempts and redistribute marks between the parts as you choose, provided all markers agree. Indeed, to a large extent, you may do as you choose, but of course you must be consistent across all the scripts. The students will eventually see a copy of this sheet.*

P.S. If you're wondering why the result is true, consider a contour integral of P/Q round a large circle in the complex plane!]