Question 3. The function $I(m,\, n)$, where $m \geqslant 0$ and $n \geqslant 0$ are integers, is defined by

$$I(m,\, n) = \int_0^1 x^m \left[-\log x\right]^n \, dx.$$

Use integration by parts to obtain for $n > 0$, a relation of the form

$$I(m,\, n) = kI(m,\, n-1), \qquad \text{where } k \text{ is to be found.}$$

Hence obtain an explicit formula for $I(m,\, n)$, and show that $I(5,\, 4) = \frac{1}{324}$.

Answer. We note $d/dx[-\log(x)]^n = -n/x[-\log(x)]^{n-1}$. Integrating by parts, we have

$$\int_0^1 x^m \left[-\log x\right]^n \, dx = \left[\frac{x^{m+1}}{m+1}\left[-\log x\right]^n\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\left(\frac{-n}{x}\right)\left[-\log x\right]^{n-1} \, dx.$$

The first term vanishes provided $n > 0$ as the power dominates the infinite logarithm at $x = 0$, while $\log 1 = 0$. Thus, as required,

$$I(m,\, n) = \left(\frac{n}{m+1}\right) I(m,\, n-1). \qquad \textbf{(5 marks)}$$

It follows that $I(m,\, n) = n!/(m+1)^n I(m, 0) = n!/(m+1)^{n+1}$, **(3 marks)**
and in particular

$$I(5,\, 4) = \frac{4}{6} I(5,\, 3) = \frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{1}{6}\int_0^1 x^5 \, dx = \frac{24}{6^5} = \frac{4}{6^4} = \frac{1}{324}. \qquad \textbf{(2 marks)}$$

Total : **10**

[Notes for markers: A fairly straightforward reduction formula. You will have to judge whether they have given sufficient attention to the (zero) boundary term. It is also possible to derive the result substituting $x = e^{-u}$ but I don't expect anyone to do that. If all markers agree you can redistribute marks between the parts. Indeed, you may to a large extent do as you choose, but of course you must be consistent across all the scripts. Remember they only have 15 minutes for this question. The students will eventually see a copy of this sheet. Don't forget to initial your work to help in the "Meet your Marker" sessions.]