Question 3.
In terms of polar coordinates (r, θ) centred on the sun, the (planar) path of a spaceship is given by

$$\frac{l}{r} = 1 + e\cos\theta,$$

where l and e are positive constants.
(a) Find the maximum and minimum values attained by r provided a certain condition holds, and deduce that the spaceship is trapped between two circles

$$r_{min} \leqslant r \leqslant r_{max}.$$

(b) If the positive x-axis corresponds to $\theta = 0$, show that

$$\frac{(x-d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

provided a condition holds, and find the constants d, a, b in terms of l and e.

Answer. (a) Clearly $1 - e \leqslant 1 + e\cos\theta \leqslant 1 + e$, and so provided $1 - e > 0$,

$$r_{min} = \frac{l}{1+e} \leqslant r \leqslant \frac{l}{1-e} = r_{max}. \qquad \textbf{(2 marks)}$$

If $e \geqslant 1$, then there is a value of θ such that $1 + e\cos\theta = 0$ and $r \to \infty$ (no maximum). So we require $e < 1$. **(2 marks)**
(b) We have $x = r\cos\theta$ and $r^2 = x^2 + y^2$. Now $l = r + er\cos\theta = r + ex$. So

$$(l - ex)^2 = r^2 = x^2 + y^2 \quad \Longrightarrow \quad x^2(1-e^2) + 2elx + y^2 = l^2.$$

Thus

$$\left(x + \frac{el}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{l^2}{1-e^2} + \frac{e^2l^2}{(1-e^2)^2} = \frac{l^2}{(1-e^2)^2}. \qquad \textbf{(3 marks)}$$

So dividing through, we have $(x-d)^2/a^2 + y^2/b^2 = 1$, provided.

$$d = \frac{-el}{1-e^2}, \qquad a = \frac{l}{1-e^2}, \qquad b = \frac{l}{\sqrt{1-e^2}}. \qquad \textbf{(3 marks)}$$

Once again, we require $e < 1$ for this to make sense. **Total 10**
[Notes for markers: For part (b) give some credit for well motivated but inaccurate algebra. Common errors will be the minus sign in d and confusing l and 1. Some may express x and y parametrically in terms of θ and then substitute in the equation and show that it works for suitable d, a and b. That is acceptable if well presented. Remember that they have under 15 minutes for the question. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will see a copy of this sheet.]