**Name (IN CAPITAL LETTERS!):** ............................ **TID:**

**CID:** ............................ **Personal tutor:** ............................

**Question 1.**
The function $f(x)$ is defined for $x > 0$ by

$$f(x) = x^{g(x)} \qquad \text{where} \quad g(x) = x^x.$$

(a) As $x$ decreases to zero through positive values, find the limit

$$\lim_{x\to 0+} f(x).$$

(b) Using any method, find $f'(x)/[f(x)g(x)]$, where $f'$ denotes the derivative.
(c) Estimate the limit as $x \to 0+$ of $f'(x)$ (only sketchy justification required).

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**Answer.** (a) First consider

$$\lim_{x\to 0+} g(x) = \lim_{x\to 0+} e^{x\log x} = e^0 = 1. \qquad \textbf{(2 marks)}$$

Thus

$$\lim_{x\to 0+} f(x) = 0^1 = 0. \qquad \textbf{(1, total 3 marks)}$$

(b) Again, we write $f = \exp[\log(x)g(x)]$, so that

$$f' = [\log(x)g(x)]' f(x) = gf/x + fg'(x)\log(x).$$

Now $g = \exp(x\log(x))$ so that

$$g' = (\log x + 1)g.$$

Combining the above, we have

$$f'(x) = f(x)g(x)\left[\frac{1}{x} + \log(x)\left(\log x + 1\right)\right] = x^x x^{x^x}\left[\frac{1}{x} + \log(x)\left(\log x + 1\right)\right]$$

**(5 marks)**

(c) As $x \to 0$ we know $g \to 1$ so we expect $f(x) = x^g \simeq x$ for small $x$. So we expect

$$\lim_{x\to 0+} f'(x) = 1. \qquad \textbf{(2 marks)}$$

**Total 10**

*[Notes for markers: This could well be a messy question to mark – sorry! They may perform some operations on limits without formal justification, as I have above. There may be a few dreadful attempts at differentiating $x^x$. Remember that they have under 15 minutes for the question. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will see a copy of this sheet.]*