Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 4.
A professor once told me he gave up applied Maths because he was terrified of Bessel functions. One of these harmless functions is defined as an infinite series by

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \ldots = \sum_{n=0}^{\infty} \frac{(-x^2/4)^n}{(n!)^2}.$$

(1) Use the ratio test to determine the radius of convergence of this series.
(2) Assuming it is legitimate to differentiate the series term by term, show that $x^2 J_0'' + x J_0' = -x^r J_0$, for some value of r, where $'$ denotes a derivative.

Answer. Defining the n^{th} term of the series as $u_n = (-x^2/4)^n/(n!)^2$, we have

$$L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{-x^2/4}{(n+1)^2} \right| = 0$$

As $L < 1$ for every value of x, the radius of convergence is infinite. **(5 marks)**
(2) If we write $J_0 = \sum a_n x^{2n}$, then

$$x^2 J_0'' + x J_0' = \sum [2n(2n-1) + 2n] a_n x^{2n} = \sum 4n^2 a_n x^{2n} = \sum \frac{4(-x^2/4)^n}{[(n-1)!]^2}.$$

So writing $m = n - 1$, we have

$$x^2 J_0'' + x J_0' = \sum (-x^2) \frac{(-x^2/4)^m}{(m!)^2} = -x^2 J_0(x).$$ **(5 marks)**

Total 10

[Notes for markers: This could well be a messy question to mark – sorry! Obviously do not give full credit in part 2 unless they consider the general term, and cancel n into the factorial. In part 1 they may come unstuck for not taking the limit as $n \to \infty$*, or for quoting some formula for* $\sum a_n x^n$ *and then having* $a_n = 0$ *for odd n. Remember that they have under 15 minutes for the question. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will see a copy of this sheet.]*