Name (IN CAPITAL LETTERS!): ..... **TID:** ☐

CID: ..... **Personal tutor:** .....

Question 2.
By definition, the hyperbolic sine function, $\sinh x = (e^x - e^{-x})/2$.

(a) Sketch roughly the graph $y = \sinh x$. Is $\sinh x$ invertible for all x? **(2 marks)**

(b) Prove that $\sinh(3x) = 4\sinh^3 x + 3\sinh x$. **(3 marks)**

(c) Now suppose that x satisfies the cubic equation

$$x^3 + x = a.$$

Find a solution in the form $x = r\sinh t$, where r and t are to be found and may depend on a. **(5 marks)**

Answer. (a) For graph 1 mark (given in lectures). From the increasing nature of the function, a unique inverse function exists (1 mark). **(2 marks)**

(b) From the definition,

$$4\sinh^3 x = 4\left[\frac{e^x - e^{-x}}{2}\right]^3 = \frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{2} = \sinh(3x) - 3\sinh x,$$

and the result follows. **(3 marks)**

(c) Substituting, we have

$$r^3\sinh^3 t + r\sinh t = a \qquad \longrightarrow \qquad 3r^2\sinh^3 t + 3\sinh t = 3a/r$$

If we choose $3r^2 = 4$, then this equation becomes

$$\sinh 3t = 3a/r \qquad \longrightarrow \qquad 3t = \sinh^{-1}(3a/r)$$

Thus a solution is

$$x = \frac{2}{\sqrt{3}}\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{3a\sqrt{3}}{2}\right)\right]. \qquad \textbf{(5 marks)}$$

Total 10

[Notes for markers: As ever, we are interested as much in coherent reasoning as in the actual answers. There may be other ways of approaching the questions, whose worth you should assess yourself. Do not give extra credit for using the log formula for $\sinh^{-1}$*, but you may have to check whether they have got it right. I expect there will be several arithmetical slips in deriving the final answer – do not be too harsh on those. Remember that they have under 15 minutes for the question. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will eventually receive a copy of this sheet.]*