Question 4.
Eigenvalues are far too important to be left in the sole care of even Prof Liebeck. λ is called an eigenvalue of the differential operator $-d^2/dx^2$ if there is a non-zero solution $y(x)$ to the equation

$$-y'' = \lambda y, \qquad \text{with } y(0) = 0 = y(1).$$

(a) **Without solving this equation,** show that

$$\lambda \int_0^1 y^2\,dx = \int_0^1 (y')^2\,dx. \tag{$*$}$$

(b) Equation $(*)$ can be used to find good approximations to λ. Approximating y by a quadratic $u(x)$ which is zero at both $x = 0$ and $x = 1$, use $(*)$ to obtain an estimate for one possible value of λ.

[N.B. For checking purposes only, the exact answer you are approximating is π^2.]

Answer. (a) Multiplying the equation by y and integrating between 0 and 1,

$$\lambda \int_0^1 y^2\,dx = \int_0^1 -yy''\,dx = \Big[-yy'\Big]_0^1 + \int_0^1 (y')^2\,dx, \qquad \textbf{(3 marks)}$$

integrating by parts. Now as $y = 0$ at both endpoints, the result follows. **(2 marks)**
(b) A quadratic which is zero at $x = 0$ and $x = 1$ takes the form $u = Ax(1-x)$, for which $u' = A(1-2x)$. Then w.l.o.g. setting $A = 1$, we have

$$\int_0^1 (u')^2\,dx = \int_0^1 (1 - 4x + 4x^2)\,dx = 1 - 2 + \tfrac{4}{3} = \tfrac{1}{3}, \qquad \textbf{(2 marks)}$$

while

$$\int_0^1 u^2\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \tfrac{1}{3} - \tfrac{2}{4} + \tfrac{1}{5} = (10 - 15 + 6)/30 = \tfrac{1}{30}. \textbf{(2 marks)}$$

Thus the estimate is $30/3 = 10$. **(1 marks)**
Note – before the advent of computers, this was a useful method to find eigenvalues of difficult problems. It gives a surprisingly good approximation to the exact solution with the smallest λ which here is $y = \sin(\pi x)$, $\lambda = \pi^2$ **Total 10**
[Notes for markers: If anyone ignores instructions and solves the ODE, award them some credit. Do not be too severe on arithmetical slips if the method is clearly presented. Remember that they have under 15 minutes for the question. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will see a copy of this sheet.]