Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 3. The Integral Test for convergence of infinite series
The function $f(x) > 0$ for all x, and $f(x)$ is a decreasing function and is integrable. By considering rectangles in a clear diagram or otherwise, show that

$$\sum_{n=2}^{N} f(n) < \int_1^N f(x)\,dx < \sum_{n=1}^{N-1} f(n).$$

Deduce that the infinite series

$$\sum_{n=1}^{\infty} f(n) \quad \text{converges if and only if} \quad \int_1^\infty f(x)\,dx \quad \text{converges.}$$

Using this result, determine whether or not the following series converge:

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n} \qquad (b) \quad \sum_{n=2}^{\infty} \frac{1}{n\log n} \qquad (c) \quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

(You may assume "obvious" results such as an increasing sequence either tends to a limit or goes to infinity.)

Answer. Proof: If x is not an integer and k is the largest integer less than x, we have $k < x < k+1$ and as f is decreasing

$$f(k) > f(x) > f(k+1) \quad \Longrightarrow \quad \int_k^{k+1} f(k)dx > \int_k^{k+1} f(x)dx > \int_k^{k+1} f(k+1)dx$$

Thus

$$f(k) > \int_k^{k+1} f(x)dx > f(k+1) \quad \Longrightarrow \quad \sum_{k=1}^{N-1} f(k) < \int_1^N f(x)\,dx < \sum_{n=2}^{N} f(n)$$

using the additive properties of integrals, as required. **(3 marks)**

[N.B. A good diagram is acceptable proof here, provided the areas covered by the sums of rectangles are clearly shown.]

Now let $N \to \infty$. If the integral is finite then from the left inequality the sum of the series is bounded above, and as all terms are positive it must converge. If conversely, the integral tends to infinity, then from the rightmost inequality so must the series. **(2 marks)**

[N.B. This can be argued more than one way. However, if they don't recognise that 'if and only if' requires two way proofs, deduct a mark]

(a) To consider $\sum \frac{1}{n}$, we look at $\int dx/x = \log x$, noting that $1/x$ is a positive decreasing function as required. This diverges as $x \to \infty$, so the infinite integral does not exist. Thus the series diverges also. **(1 mark)**

(b) Again, $x \log x > 0$ and is increasing, so the test conditions hold. Now

$$\int \frac{dx}{x \log x} = \int \frac{du}{u} = \log u \qquad \text{putting } u = \log x$$

This also goes to infinity as $x \to \infty$, so this series also does not converge. **(1 mark)**

(c) Again the function is decreasing and positive, but now

$$\int \frac{dx}{x(\log x)^2} = \int \frac{du}{u^2} = -u^{-1} \to 0$$

as $x \to \infty$. Thus this integral exists, and so the infinite series converges. **(2 marks)**

We note that our series start at $n = 2$ and not $n = 1$ (because $\log 1 = 0$). A single term does not affect whether an infinite series converges or not. Formally, we could for example define $f(x) = f(2) - (x - 2)$ for $x < 2$ or we can write $m = n - 1$ in the sums before starting, or we can say obviously the same arguments we used to prove the original results obviously go over if we start at $x = 2$ not $x = 1$. Even stating "We will assume that a similar result holds if we start at $n = 2$" is good enough for **(1 mark)**

Total 10

To marker: An exercise in semi-rigorous analysis... Demand clarity of argument rather than rigour. They can say things are obvious provided they are clear about what the 'things' are. Thus it is fine to say that the area under the curve is obviously less that the area of the larger rectangles etc. Giove marks for recognising what results need proving rather than requiring proofs... as ever, use your judgement.