Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 2. We showed in lectures that the radius of curvature, ρ, of the curve $y = f(x)$ is

$$\rho = \frac{(1 + f'^2)^{3/2}}{f''}.$$

Suppose the curve is given parametrically, in the form $x = F(t)$, $y = G(t)$. Obtain a simple formula for $\rho(t)$ in terms of the derivatives of F and G.

Calculate ρ when $F = 1 + \cos t + \sin t$ and $G = 1 - \cos t + \sin t$.

Hence or otherwise sketch the curve.

Answer. We have $dy/dx = G'/F'$, and so

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{G'}{F'}\right)\frac{dt}{dx} = \frac{F'G'' - F''G'}{F'^2}\frac{1}{F'}.$$

Thus

$$\rho = \frac{(1 + G'^2/F'^2)^{3/2}}{(F'G'' - F''G')/F'^3} = \frac{(F'^2 + G'^2)^{3/2}}{(F'G'' - F''G')}. \qquad \textbf{(5 marks)}$$

When $F = 1 + \cos t + \sin t$, we have $F' = \cos t - \sin t$ and $F'' = -(\sin t + \cos t)$ and similarly $G' = \sin t + \cos t$ and $G'' = \cos t - \sin t$. So $F'^2 + G'^2 = 2\cos^2 t + 2\sin^2 t = 2$. Furthermore,

$$(F'G'' - F''G') = (\cos t - \sin t)^2 + (\cos t + \sin t)^2 = 2. \qquad \textbf{(3 marks)}$$

Putting all this together, we have $\rho = 2^{3/2}/2 = \sqrt{2}$, a constant. Therefore the curve is a circle of radius $\sqrt{2}$. If t increases by π, both $(x - 1)$ and $(y - 1)$ change sign, so the curve is symmetric about $(1, 1)$, so that is its centre. The curve is a circle of radius $\sqrt{2}$, centre $(1, 1)$ and passes through the origin. **(2 marks)**

Alternatively, one could argue that $x = 1 + \sqrt{2}\sin(t + \frac{1}{4}\pi)$, $y = 1 + \sqrt{2}\cos(t + \frac{1}{4}\pi)$ and so $(x - 1)^2 + (y - 1)^2 = 2$ and so on.

Total 10

To marker: Some people try very hard to avoid learning new things, and so will try to eliminate t from the parametric definitions. This actually works well for this example – maybe I should have chosen a harder curve! If they get the curvature and graph correct, award full marks for that, but unless they obtain the general parametric form of ρ*, they have not done as instructed, and so should be penalised in the first part. A description of the curve (rather than a sketch) should suffice for two marks, but unless they show it is a circle somehow, only award 1. If they don't simplify the final formula by multiplying up by* F'^3 *deduct 1. As ever, use your judgement. You may alter the mark scheme if you wish, provided you do it consistently, of course!*