Name (IN CAPITAL LETTERS!): .... **TID:**

CID: .... **Personal tutor:** ....

Question 1. Consider the cubic function, for non-zero constants a and c

$$f(x) = x^3 + 3ax^2 + c.$$

(a) Find the points where $f'(x) = 0$ and the corresponding function values.
(b) By considering these local minimum and maximum points, find the condition on a and c for $f(x)$ to have 3 distinct real roots.
(c) Considering the cases of positive and negative a and c separately, find the conditions on a and c for $f(x)$ to have at least one positive root.

[Note you may use graphical arguments, provided these are clearly explained.]

Answer. (a) We have

$$f' = 3x^2 + 6ax = 0 \qquad \text{when } x = 0 \text{ or } x = -2a.$$

Now $f(0) = c$ and $f(-2a) = c + 4a^3$ **(2 marks)**
(b) The issue is complicated because we don't know whether $a > 0$ or $a < 0$, but we do know the general shape of the curve. f' is positive except between the two roots. Thus as x increases from $-\infty$, f increases to a maximum, then decreases to a minimum, then increases to $+\infty$.

By considering the shape of the graph, we will have 3 real roots if the maximum value is positive and then the minimum value is negative. Now from part (a) $f(0) = c$ and $f(-2a) = c + 4a^3$. Thus there will be 3 real roots iff the product of these two is negative, or

$$c(c + 4a^3) < 0 \qquad \textbf{(4 marks)}.$$

Note this can be written in different ways, for example "Either $-4a^3 > c > 0$, or $4a^3 > -c > 0$." Often only one of these will be given, for half marks.
(c) If $a > 0$ and $c > 0$, then if $x > 0$ all 3 terms are positive, and $x^3 + 3ax^2 + c > 0$ so no root exists.
If $c < 0$, then $f(0) < 0$ and since $f \to \infty$ as $x \to \infty$, by continuity, there must be a root in $x > 0$. So $c < 0$ is **sufficient** for a positive root.
If $a < 0$ and $c > 0$, then f has a positive maximum at $x = 0$, and then a minimum at $x = -2a$. Thus f will have a positive root iff $f(-2a) < 0$, from the shape of the graph.

Now $f(-2a) = c + 4a^3$. So if $0 < c < -4a^3$, then $f(x)$ will have a root somewhere in $x > -2a(> 0)$.
So a positive root exists if either $c < 0$ **(2 marks)** or $0 < c < -4a^3$ **(2 marks)**.

Total 10

To marker: This could be tricky to mark, as a number of arguments involving graphs and signs of gradient will be produced. Be specially vigilant of implicit assumptions that $a > 0$. If some one misevaluates f(-2a) for example, you may still award most of the credit if the (harder) arguments of parts (b) and (c) are good. There may be other arguments produced, whose worth you will have to assess yourself... good luck!