Name (IN CAPITAL LETTERS!): .... **TID:**

CID: .... **Personal tutor:** ....

Question 4. Calculate the three limits as $x \to \infty$ of

$$(a) \quad \frac{\log(1+x)}{x} \qquad (b) \quad x\left(\sqrt{2+x^2}-\sqrt{1+x^2}\right) \qquad (c) \quad (\tanh x)^{\cosh(2x)}.$$

Answer.

(a) As $x \to \infty$, exponentials dominate powers which dominate logarithms. Now $\log(1+x)/x = \log x/x + O(1/x^2)$, and $\lim \log x/x = 0$. [Or write $(1+x) = e^t$ and quote that $\lim_{t\to\infty} t/(e^t-1) = 0$.] (Stating "powers beat logs" is enough.) **(2 marks)**

(b) We first manipulate the expressions so that we can expand in $1/x$, not x:

$$x\left(\sqrt{2+x^2}-\sqrt{1+x^2}\right) = x^2\left[\left(\frac{2}{x^2}+1\right)^{1/2} - \left(\frac{1}{x^2}+1\right)^{1/2}\right]$$

$$= x^2\left[1+\frac{2}{2x^2}-1-\frac{1}{2x^2}+O(x^{-4})\right] = \tfrac{1}{2}+O(x^{-2}) \qquad \text{so limit is } \tfrac{1}{2}. \qquad \textbf{(3 marks)}$$

(c) As $x \to \infty$, $\tanh x \to 1$, and $\cosh 2x \to \infty$. More precisely, $\cosh 2x \simeq \frac{1}{2}e^{2x}$ and

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = (1-e^{-2x})(1+e^{-2x})^{-1} = 1 - 2e^{-2x} + O(e^{-4x})$$

Taking logarithms, we have

$$\log\left[\tanh x^{\cosh 2x}\right] = \cosh 2x \log(\tanh x) \simeq \tfrac{1}{2}e^{2x}\left(-2e^{-2x}\right) = -1$$

Thus exponentiating, we have

$$\lim_{x\to\infty}\left[\tanh x^{\cosh 2x}\right] = e^{-1}. \qquad \textbf{(5 marks)}$$

Total 10

To markers: As ever, you may award partial credit for sensible partial answers. You do not have to give full credit for the correct answers if badly argued or explained. However, rigorous justification is not required – we want them to develop a **feel** *for how functions behave.*