Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 2.
(a) The function $f(x)$ is defined for all x by the infinite series

$$f(x) = \tfrac{1}{2} - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 \ldots = \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$$

By grouping the terms of the series in two different ways, in a similar manner to Friday's lecture, infer that $f(x)$ has a root in the interval $(\frac{1}{2}, 1)$.
(b) Relate $f(x)$ to the exponential function and deduce the exact value of the root.

Answer. First write the series in the form

$$f(x) = \left(\tfrac{1}{2} - x\right) + \tfrac{1}{2}x^2\left(1 - \tfrac{1}{3}x\right) + \tfrac{1}{24}x^4\left(1 - \tfrac{1}{5}x\right) + \ldots$$

If $x < \frac{1}{2}$, all the brackets are positive so we that $f(x) > 0$ for $x \leqslant \frac{1}{2}$. **(2 marks)**

Now write the series as

$$f(x) = \tfrac{1}{2}(1 - 2x + x^2) - \tfrac{1}{6}x^3\left(1 - \tfrac{1}{4}x\right) - \tfrac{1}{5!}x^5\left(1 - \tfrac{1}{6}x\right) - \ldots$$

All the brackets bar the first are positive for $x < 4$. The first vanishes if $x = 1$, so we conclude that $f(1) < 0$. **(3 marks)**

As the function changes sign between $x = 1/2$ and $x = 1$, it must pass through zero somewhere between the two (assuming $f(x)$ is continuous). We conclude that $f(a) = 0$ for some a in $1/2 < a < 1$ **(1 mark)**

(b) From the definition of the exponential function, we have

$$f(x) = \exp(-x) - \tfrac{1}{2}. \qquad \textbf{(2 marks)}$$

So $f(x) = 0$ if $-x = \log(1/2) = -\log 2$. Thus the real root is $x = \log 2$. **(2 marks)**

Total 10

[Notes for markers: Have a look at the 1st page of the handout on the webpage about the 1st zero of cosine. This was given to them and gone through in lectures on Friday, and they should be able to reproduce the argument with more or less the same level of rigour. It is up to you to assess how clearly they explain the steps in the argument – you need not give full marks to a correct final answer which is not clearly explained. Likewise, you may, if you wish, penalise poor presentation. Remember that they have under 15 minutes for the question, though. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will eventually receive a copy of this sheet.]