Question 1.
(a) An M1F lecture lasts between $t = 0$ and $t = 1$. The proportion of the class which is unhappy at any given time is given by the function

$$h(t) = \frac{2t}{1 + t^2}.$$

Prof Thomas likes to ask a "phone" question when $h(t) = a$, where $1 \geqslant a \geqslant 0$. Find the value of t when this happens as a function of a.
(b) Prof Liebeck lectures an hour later between $t = 1$ and $t = 2$, with the same unhappiness function, $h(t)$. What value of t should he choose to give $h(t) = a$?
(c) Which values of a can Prof Liebeck not obtain, no matter how well he lectures?

Answer. (a) Solving $a = 2t/(1 + t^2)$, we have

$$t = \frac{1 \pm \sqrt{1 - a^2}}{a}. \qquad \textbf{(3 marks)}$$

We must have $0 \leqslant a \leqslant 1$ and we want to ensure that $0 \leqslant t \leqslant 1$. If we choose the $+$ sign we would have $t \geqslant 1/a \geqslant 1$, so that doesn't work. We conclude that

$$t = \frac{1 - \sqrt{1 - a^2}}{a}. \qquad \textbf{(2 marks)}$$

(Alternatively sketch the curve $y = h(x)$ and observe that for a given y the smaller value of x lies in $(0, 1)$.)
(b) This time we want $t > 1$ so we choose

$$t = \frac{1 + \sqrt{1 - a^2}}{a}. \qquad \textbf{(2 marks)}$$

(c) However, when a is small, t becomes too large, and the lecture ends at $t = 2$. Differentiating, we have

$$h'(t) = \frac{2(1 - t^2)}{(1 + t^2)^2} < 0 \qquad \text{for} \quad t > 1.$$

Thus the unhappiness decreases steadily throughout the lecture from $h = 1$ at $t = 1$ to $h = 4/5$ when $t = 2$. We conclude that he cannot reach any $a < 4/5$. **(3 marks)**

Total 10

[Notes for markers: As ever, we are as interested in coherent reasoning as in the actual answers. There may be other ways of approaching the questions, whose worth you should assess yourself. Give credit for clarity of argument – you need not give full marks to a correct final answer which is not clearly explained. Likewise, you may, if you wish, penalise poor presentation. Remember that they have under 15 minutes for the question, though. You may, to a large extent, do as you choose, but of course you must be consistent across all the scripts. The students will eventually receive a copy of this sheet.]