Name (IN CAPITAL LETTERS!): .... **TID:** [ ]

CID: .... **Personal tutor:** ....

Question 4. *After getting sunburned on holiday, Prof Thomas answered a Small Ad for a "tan reduction formula." Sadder but wiser, he received the following question:*

$$\text{We define, for natural numbers } n, \qquad I_n = \int_0^{\pi/4} \tan^n x \, dx.$$

(a) Using a relation between $\sec x$ and $\tan x$ or otherwise, show that

$$I_n = \frac{1}{n-1} - I_{n-2} \quad \text{for } n > 1.$$

(b) Deduce an expression for I_{4k} where the integer $k > 0$.
(c) Sketch the integrand $\tan^n x$ over the range of the integral for increasing values of n, and infer the limit of I_n as $n \to \infty$.
(d) Hence express π as an infinite series.

Answer. (a) We have $\tan^2 x = \sec^2 x - 1$, and so

$$I_n = \int_0^{\pi/4} \sec^2 x \tan^{n-2} x \, dx - I_{n-2} = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2}.$$

(3 marks)

(b) By repeated application,

$$I_{4k} = \frac{1}{4k-1} - \frac{1}{4k-3} + I_{4k-4} = \frac{1}{4k-1} - \frac{1}{4k-3} + \ldots + \tfrac{1}{3} - 1 + I_0$$

Now $I_0 = \pi/4$, and so

$$I_{4k} = \tfrac{1}{4}\pi - \left[1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots + \frac{1}{4k-1}\right].$$

(2 marks)

(c) For $0 < x < \pi/4$, we have $0 < \tan x < 1$. Thus $\tan^n x \to 0$ as $n \to \infty$. However $\tan^n(\pi/4) = 1$ for all n. In the limit, the integrand is discontinuous, but for large but finite n it is very small except near $x = \frac{1}{4}\pi$, where is rises steeply towards 1. The area under this curve is small – the integrand is very small except in a very

thin region where is is bounded. For suitable sketch and arguments, **(3 marks)**
(d) Since $I_n \to 0$ as $n \to \infty$, we deduce that $I_{4k} \to 0$ as $k \to \infty$ and hence that

$$\pi = 4\left[1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + ...\right] = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{2n-1}.$$ **(2 marks)**

Total 10

To marker: As ever, feel free to adjust the mark scheme consistently, or to award bonus marks for good maths, or to deduct marks as appropriate. There are other equivalent ways of deriving part (a). They do not need to give the general term in the infinite series, provided it's done clearly. But they ought to indicate the significance of $n = 4k$ *rather than* $n = 2k$*, which hits 0 with the opposite parity, so that the formula is essentially negative.*