(1) A **vector** is a quantity with both **magnitude** and **direction**. In contrast, a **scalar** is just a number, with magnitude only. Usually we represent a vector in terms of its **components** with respect to cartesian axes, either as a **row vector**,

$$\mathbf{a}^T = (a_1,\ a_2,\ a_3) \qquad \text{or a } \textbf{column vector} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} .$$

The suffix ^{T}, denotes the matrix **Transpose** and is often left out. Vectors will be written either in bold ($\mathbf{a}$) or underlined ($\underline{a}$).

(2) The **dimension** of a vector is the number of components it has. In this course we consider mainly 3-dimensional vectors, which we relate to ordinary three-dimensional space. The vector $\mathbf{a}$ can be thought of as pointing from the origin to the point A whose cartesian coordinates are $(a_1,\ a_2,\ a_3)$. Such a vector is called the **position vector** of A.

(3) To add two vectors, we simply add the corresponding components:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3) .$$

Similarly, we can subtract two vectors. Geometrically, we can represent $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ as the sides of a triangle. Similarly, we can subtract two vectors. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of two points A and B, then $\mathbf{b} - \mathbf{a}$ is the vector **displacement** from A to B.

To multiply a vector by a scalar, λ, we just multiply each component in turn:

$$\lambda\mathbf{a} = (\lambda a_1,\ \lambda a_2,\ \lambda a_3) .$$

(4) The **magnitude**, length, or **modulus** of a vector $\mathbf{a}$ is written $|\mathbf{a}|$ and pronounced "mod a". In terms of the components

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} .$$

A vector $\mathbf{a}$ for which $|\mathbf{a}| = 1$ is called a **unit vector**. We shall use a circumflex to denote a unit vector, e.g. $\widehat{\mathbf{n}}$.

(5) The **scalar product** or **dot product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written $\mathbf{a} \cdot \mathbf{b}$ and pronounced "a dot b." It is a **scalar**, given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 . \qquad \text{Alternatively,} \qquad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta ,$$

where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$. Note:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}, \qquad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}, \qquad \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 .$$

If $\widehat{\mathbf{n}}$ is a unit vector, then $\mathbf{a} \cdot \widehat{\mathbf{n}}$ is the **component** of $\mathbf{a}$ in the **direction of** $\widehat{\mathbf{n}}$.

(6) Two vectors are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. In general the angle θ between two vectors may be found from their components by using

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \ .$$

(7) The **vector product**, or **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is written $\mathbf{a} \wedge \mathbf{b}$ or $\mathbf{a} \times \mathbf{b}$ and pronounced "a cross b". It is defined only for 3-dimensional vectors. It is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so that

$$\mathbf{a} \cdot (\mathbf{a} \wedge \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0 \ .$$

Its modulus is given by

$$|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \ ,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The vectors $\mathbf{a}$, $\mathbf{b}$ and $(\mathbf{a} \wedge \mathbf{b})$ form a **right-handed system**. This means that if $\mathbf{a}$ and $\mathbf{b}$ point respectively along the x-axis and y-axis in the usual configuration of a graph, then $(\mathbf{a} \wedge \mathbf{b})$ points up out of the graph-paper/whiteboard. In terms of components,

$$\mathbf{a} \wedge \mathbf{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1) \ .$$

This formula can also be written as a determinant. Note that

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}, \qquad \mathbf{a} \wedge \mathbf{a} = 0, \qquad \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c} \ .$$

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **parallel** (that is, they point in the same or opposite directions) if and only if $\mathbf{a} \wedge \mathbf{b} = 0$.

(8) A quantity such as $(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}$ is called a **scalar triple product**. It is equal to zero if and only if the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ lie in the same plane, in which case they are said to be **linearly dependent**. Otherwise, they are **linearly independent**. It can be shown that $(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$ and $(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \wedge \mathbf{a}) \cdot \mathbf{b}$.

(9) A quantity such as $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ or $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ is called a **vector triple product**. Note that these two quantities are **not usually the same**; the brackets are important. It can be shown that

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$$
$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \ .$$

Always use brackets unless you are certain there is no ambiguity.

(10) Vectors may be differentiated. Assuming the coordinate axes remain fixed, then the components are differentiated independently.

$$\frac{d\mathbf{a}}{dt} = \left(\frac{da_1}{dt}, \frac{da_2}{dt}, \frac{da_3}{dt} \right) \ .$$

If, however, the axes change in time, then this must be taken into account (see later.) In this course we shall sometimes use a dot to denote a time derivative, $\dot{\mathbf{a}} \equiv d\mathbf{a}/dt$. Note that

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \dot{\mathbf{a}} \cdot \mathbf{b} + \mathbf{a} \cdot \dot{\mathbf{b}} \qquad \text{and} \qquad \frac{d}{dt}(\mathbf{a} \wedge \mathbf{b}) = \dot{\mathbf{a}} \wedge \mathbf{b} + \mathbf{a} \wedge \dot{\mathbf{b}} \ .$$