M1A1: Problem Sheet 4 Oscillators & circular motion

(Throughout this sheet, m denotes an appropriate mass.)

1. In lectures we showed that if $k > \omega > 0$, and $\ddot{x} + 2k\dot{x} + \omega^2 x = 0$, then the decay rate of x was $k_1 = k - \sqrt{k^2 - \omega^2}$. Prove that $k_1 < \omega$, and explain the significance of this result.

2. An undamped linear oscillator is acted on by a force $F(t)$ so that

$$\ddot{x} + \omega^2 x = \frac{1}{m} F(t) = \begin{cases} F_0/m & \text{for} \quad 0 < t < T \\ 0 & \text{for} \quad T < t \end{cases}$$

where F_0 and T are constants. Find $x(t)$ given that $x(0) = 0 = \dot{x}(0)$, and hence show that for $t > T$ the amplitude C of oscillation is

$$C = \frac{2F_0}{m\omega^2} \sin \tfrac{1}{2}\omega T \; .$$

3. Calculate the total work done by the force $F(t)$ during the motion in question 2, and compare it with the total energy (kinetic and potential) for $t > T$.

4. A lecturer attaches a piece of elastic to a cup, and holds the other end a height y above a table. The elastic has a natural length L and a spring constant λ. If $y = y_0$, a constant, find the equilibrium height x_0 of the cup.
He now forces one end of the elastic to move with $y = y_0 + a\cos\omega t$. Find the equation satisfied by the cup height, $x(t)$, when a linear frictional force $-2mk\dot{x}$ acts on the cup.
If $k = 0$, what is the resonant frequency of the system? Find $x(t)$ if ω takes this resonant value, given that at $t = 0$ the cup is at rest in its position of static equilibrium.

5. Following a skiing accident, our winter sportsman walks on crutches. He develops a technique whereby he pushes off from the ground with his good leg, and until this leg again touches the ground, his centre of mass moves in a circle of radius h = 1m. The crutches can be regarded as a single crutch rotating in a vertical plane supporting the centre of mass.

Show that the vertical reaction, N, between the crutch and the (rough) ground is

$$N = m(g\cos^2\theta - h\dot{\theta}^2\cos\theta)$$

where θ is the angle the crutch makes with the vertical.

As he becomes adept on crutches, he manages to achieve a forward speed of 3.5 m/s. What happens?