M1A1: Problem Sheet 3 1-D motion; Potentials

(Throughout this sheet, m denotes an appropriate mass.)

1. A particle is projected vertically upwards under gravity with speed u from the origin at $t = 0$. When travelling with speed v it experiences a quadratic drag $-m\mu|v|v$, where μ is constant. Formulate the problem for $v(x)$, where x is the vertical distance.
Show that when the particle returns to $x = 0$, its speed is given by

$$v^2 = \frac{gu^2}{g + \mu u^2} \ .$$

What is the terminal velocity of the particle?

2. A particle moves in $x > 0$ under the potential

$$V(x) = V_0 \left(\frac{a^2}{x^2} - \frac{a}{x} \right) \qquad \text{where} \quad V_0 > 0 \, .$$

Show that there is a position of stable equilibrium, and find the frequency of small oscillations about it.
What is the escape velocity for a particle at the equilibrium point?

3. A particle with speed u at $x = 0$ is acted upon by the potential

$$V(x) = \begin{cases} mfx & \text{for} \quad x > 0 \\ \frac{1}{2}m\omega^2 x^2 & \text{for} \quad x < 0 \ , \end{cases}$$

where ω and f are constant. Why does the formula for the frequency of small oscillations from lectures not apply? Calculate the motion exactly, and show that the particle oscillates about $x = 0$ with a period given by $T = \pi/\omega + 2u/f$.

4. Somewhat bruised, the abseiler from Q3 on Problem Sheet 2, decides to attach his harness at his centre of mass. If ϕ is the angle from the harness to his feet to the top of the cliff, show that his potential energy is

$$V(\phi) = -mg\sqrt{L^2 - h^2 \sin^2 \phi} \quad (L > h) \ ,$$

where L is the length of the rope, and h the distance from his feet to his centre of mass. Discuss the stability of the possible equilibria.

5. Even more bruised and battered, our intrepid winter sportsman decides to take up skiing. The ground takes the shape $y = f(x)$ where f is a known function and y is the vertical height. Express $\dot{y}$ and $\ddot{y}$ in terms of f, $\dot{x}$ and $\ddot{x}$.
Starting from rest at $y = H$, he skis along the ground in the (x, y)-plane to the top of a small hill at $x = a$, where $f(a) = h < H$, $f'(a) = 0$ and $f''(a) = -c < 0$.
Show that the vertical reaction at this point, N, is

$$N = m(\ddot{y} + g) = m(g - c\dot{x}^2) \ .$$

Assuming energy is conserved, deduce that

$$N = mg[1 - 2c(H - h)] \ .$$

What will happen if this formula predicts $N < 0$?