M1A1: Problem Sheet 1

Kinematics and Vectors

1. A particle has a time-dependent position vector

$$\mathbf{r} = (b\cos\Omega t,\ b\sin\Omega t,\ 0)\ ,$$

where b and Ω are constants. What must be the physical dimensions of these constants? Find the velocity, $\mathbf{v}$, and acceleration, $\mathbf{a}$ of the particle, and evaluate $\mathbf{r}\cdot\mathbf{r}$, $\mathbf{r}\cdot\mathbf{v}$ and $\mathbf{r}\cdot\mathbf{a}$.

2. A wide river occupies $0 < y < d$ in the $(x,\ y)$-plane. It flows faster in the middle, so that its velocity is $(V_0 y(d-y)/d^2,\ 0,\ 0)$. A woman starts swimming from the origin at time $t = 0$ with a constant velocity $(0,\ u_0,\ 0)$ **relative to the water.**

Calculate how far she has been swept downstream by the time she reaches the other bank.

3. Two particles with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ have the same acceleration,

$$\ddot{\mathbf{r}}_i = \mathbf{g} \qquad \text{for} \quad i = 1, 2\ ,$$

where $\mathbf{g}$ is a constant vector. At $t = 0$, the first particle is at the origin with velocity $\mathbf{u}_1$, while the second is at rest at position $\mathbf{r} = \mathbf{r}_0$. Find $\mathbf{r}_1$ and $\mathbf{r}_2$ for all time.

As time varies, show that the minimum distance between the particles is given by

$$|\mathbf{r}_1 - \mathbf{r}_2| = |\mathbf{r}_0|\sin\theta\ ,$$

where θ is the angle between $\mathbf{r}_0$ and $\mathbf{u}_0$.

4. A wheel occupies the region $R^2 > x^2 + z^2$ in the (x, z)-plane. It travels in the x-direction with speed v, and simultaneously rotates about its centre with angular velocity $(0,\ \omega,\ 0)$. Find the velocity at a general point $(x,\ 0,\ z)$ in the wheel.

The wheel rolls along the ground, $z = -R$. For no slipping to occur it is necessary for the velocity to be zero at the point of contact. Deduce a relation between R, v and ω.

Why can one sometimes see the spokes on the lower half of a moving bicycle wheel but not those on the top half?

5. Two particles of masses m_1 and m_2 with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ exert a force on each other. No other forces act on the particles. Assuming this interaction force is parallel to the displacement vector $(\mathbf{r}_1 - \mathbf{r}_2)$, use Newton's laws to prove that

$$\mathbf{H} = \text{constant} \qquad \text{where} \quad \mathbf{H} = m_1\mathbf{r}_1 \wedge \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \wedge \dot{\mathbf{r}}_2\ .$$

[Hint: Consider $d\mathbf{H}/dt$]