We know Newton's laws apply only in non-accelerating (inertial) frames. What happens if we use a set of Cartesian axes which are accelerating (but not rotating) with respect to an inertial frame? Suppose our origin has position vector $\mathbf{S}(t)$ with respect to an inertial frame. Then if a particle has position vectors $\mathbf{R}$ with respect to an inertial origin, and $\mathbf{r}$ with respect to the accelerating origin, we have $\mathbf{R} = \mathbf{S} + \mathbf{r}$. Now if a force $\mathbf{F}$ acts, Newton's laws require

$$\mathbf{F} = m\ddot{\mathbf{R}} \qquad \Longrightarrow \qquad \mathbf{F} - m\ddot{\mathbf{S}} = m\ddot{\mathbf{r}} \; . \tag{4.1}$$

We see therefore that we can work in the accelerating frame if we choose, provided we include an extra 'fictitious force,' $-m\ddot{\mathbf{S}}$, in the equation.

Now we know that rotation corresponds to motion in a circle which is associated with an acceleration towards the centre. If we wish to work in a rotating frame we therefore expect fictitious forces to act. This is important – we know the earth is rotating, and we need to be able to quantify the effects of this rotation on our equations.

Consider a frame (x, y, z) which is rotating about the z-axis and compare with an inertial frame (X, Y, Z). The two origins are the same for all time. We write $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ for unit vectors in the x and y-directions, and similarly for $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$. These latter two vectors are constant, but $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ vary in time. For if $\widehat{\mathbf{x}}$ makes an angle $\theta(t)$ with $\widehat{\mathbf{X}}$, then $\widehat{\mathbf{x}} = \cos\theta\,\widehat{\mathbf{X}} + \sin\theta\,\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{y}} = \cos\theta\,\widehat{\mathbf{Y}} - \sin\theta\,\widehat{\mathbf{X}}$. By calculation,

$$\frac{d}{dt}\widehat{\mathbf{x}} = \dot{\theta}\,\widehat{\mathbf{y}} \qquad \text{and} \quad \frac{d}{dt}\widehat{\mathbf{y}} = -\dot{\theta}\,\widehat{\mathbf{x}} \; .$$

Consider now any time-dependent vector $\mathbf{B} = B_1\widehat{\mathbf{x}} + B_2\widehat{\mathbf{y}}$, so that B_1 and B_2 are the components measured with respect to the rotating axes. Then the true derivative of $\mathbf{B}$ is

$$\frac{d\mathbf{B}}{dt} = \dot{B}_1\widehat{\mathbf{x}} + \dot{B}_2\widehat{\mathbf{y}} + B_1\frac{d\widehat{\mathbf{x}}}{dt} + B_2\frac{d\widehat{\mathbf{y}}}{dt} = (\dot{B}_1\widehat{\mathbf{x}} + \dot{B}_2\widehat{\mathbf{y}}) + \dot{\theta}(B_1\widehat{\mathbf{y}} - B_2\widehat{\mathbf{x}}) \; .$$

The last term we can identify as the vector product $\boldsymbol{\omega} \wedge (B_1, B_2, 0)$, where $\boldsymbol{\omega} = (0, 0, \dot{\theta})$ is the angular velocity vector.

Now $(\dot{B}_1\widehat{\mathbf{x}} + \dot{B}_2\widehat{\mathbf{y}})$ is what $d\mathbf{B}/dt$ would be if $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ were stationary, that is, it is what an observer in the rotating frame, who thinks the axes are stationary, actually measures for $\frac{d\mathbf{B}}{dt}$. We will use the suffix '*rot*' amd '*in*' to distinguish between measurements in the rotational and inertial frames. Then we have shown that

$$\boxed{\left(\frac{d\mathbf{B}}{dt}\right)_{in} = \left(\frac{d\mathbf{B}}{dt}\right)_{rot} + \boldsymbol{\omega} \wedge \mathbf{B} \; .} \tag{4.2}$$

Suppose $\mathbf{B}$ is in reality a constant vector. Then in the rotating frame, it appears to be rotating backwards with angular velocity $-\boldsymbol{\omega}$. If in (4.2) we let $\mathbf{B} = \mathbf{r}$, the position vector of a particle, we can relate the true velocity $\mathbf{v}_{in}$ and the apparent one $\mathbf{v}_{rot}$ by

$$\mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\omega} \wedge \mathbf{r} \; , \tag{4.3}$$

as we have previously obtained.

Now we define the real and apparent accelerations

$$\mathbf{a}_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} \qquad \text{and} \quad \mathbf{a}_{rot} = \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot} .$$

and set $\mathbf{B} = \mathbf{v}_{in}$ in (4.2) to obtain

$$\begin{aligned} \mathbf{a}_{in} &= \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \boldsymbol{\omega} \wedge \mathbf{v}_{in} \qquad \text{or using (4.3)} \\ &= \frac{d}{dt}\left(\mathbf{v}_{rot} + \boldsymbol{\omega} \wedge \mathbf{r}\right)_{rot} + \boldsymbol{\omega} \wedge \left(\mathbf{v}_{rot} + \boldsymbol{\omega} \wedge \mathbf{r}\right) \\ &= \mathbf{a}_{rot} + \dot{\boldsymbol{\omega}} \wedge \mathbf{r} + \boldsymbol{\omega} \wedge \mathbf{v}_{rot} + \boldsymbol{\omega} \wedge \mathbf{v}_{rot} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \end{aligned}$$

or

$$\mathbf{a}_{in} = \mathbf{a}_{rot} + \dot{\boldsymbol{\omega}} \wedge \mathbf{r} + 2\boldsymbol{\omega} \wedge \mathbf{v}_{rot} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \qquad (4.4a)$$

Since Newton's Laws apply in an inertial frame, we know that $\mathbf{F} = m\mathbf{a}_{in}$. If we choose to work in a rotating frame we should use instead

$$\mathbf{F} = m\Big[\mathbf{a} + \dot{\boldsymbol{\omega}} \wedge \mathbf{r} + 2\,\boldsymbol{\omega} \wedge \mathbf{v} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})\Big] , \qquad (4.4b)$$

where we have now written $\mathbf{v} = \mathbf{v}_{rot}$ and $\mathbf{a} = \mathbf{a}_{rot}$.

We see that when we work in a rotating frame, we should really include three extra terms in our equation! Fortunately, these terms are frequently small. We define the **Centrifugal force $\mathbf{F}_{cen}$** and **Coriolis Force**, $\mathbf{F}_{cor}$ as

$$\boxed{\mathbf{F}_{cen} = -m\,\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \qquad \text{and} \quad \mathbf{F}_{cor} = -2m\,\boldsymbol{\omega} \wedge \mathbf{v} .} \qquad (4.5)$$

For measurements on the earth, the rotation rate $\omega \equiv |\boldsymbol{\omega}| = 2\pi/(1day) \simeq 7.3 \times 10^{-5}\,s^{-1}$. The rate of variation of $\boldsymbol{\omega}$ is tiny, so we can set $\dot{\boldsymbol{\omega}} = 0$ with a clear conscience. The last term in (4.4), the centrifugal term, does have some relevance, and affects the value of g measurably. For $\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})$ is directed away from the axis of rotation of the earth, and has a magnitude $\omega^2 d$ where d is the distance from the axis, so that $d = r_e \cos\lambda$, where r_e is the radius of the earth and λ is the latitude. Thus $|\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})| \simeq 0.034 \cos\lambda \; m/s^2$ which alters the value of g. At the equator $g \simeq 9.78$ while at the poles $g \simeq 9.83$. A further adjustment in g occurs because the earth is not an exact sphere, but bulges at the equator. We shall not calculate this effect.

The apparent value of g, which we call g' is the resultant of the two vectors $\mathbf{g}$ and $\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})$. If we neglect terms proportional to ω^4, we find

$$g' \simeq g - r\omega^2 \cos^2\lambda . \qquad (4.6)$$

The Coriolis term $2\boldsymbol{\omega} \wedge \mathbf{v}$, is absolutely crucial in understanding the atmosphere and oceans. Have you ever wondered why the wind blows **along** contours of constant pressure on weather maps? As the pressure force is directed from high pressure to low pressure, one might expect air to flow **perpendicular** to the pressure contours. In fact, because of the earth's rotation, the pressure gradient is balanced by the Coriolis force, which from (4.5) is perpendicular to $\mathbf{v}$. In the Northern hemisphere, the wind blows clockwise around high pressure regions, and anticlockwise around pressure lows. In the Southern hemisphere, this behaviour is reversed.