M1A1: Solutions to Problem Sheet 5

1. The orbit equation for $u(\theta) = 1/r$ given this force is

$$u'' + u = \frac{GM}{h^2}(1 + \delta u) \qquad \text{or} \quad u'' + \omega^2 u = \frac{GM}{h^2}$$

where $\omega^2 = 1 - GM\delta/h^2$. The solution corresponding to the standard orbit equation is thus

$$\frac{l}{r} = 1 + e\cos\omega\theta \qquad \text{where} \quad l = \frac{GM}{h^2\omega^2}$$

The main difference when $\omega \neq 1$ is that r is no longer 2π-periodic in θ, so the nearly elliptical orbit doesn't quite join up each 'year' and precesses round. Currently, the earth is closest to the sun in the first week of January; this won't be the case in several thousand years.

2. Taking a frame in which the centre of mass is at rest at the origin, the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ obey $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$. Writing $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, we have $\mathbf{r} = -\mathbf{r}_1(m_1 + m_2)/m_2$, so that $r_1 \equiv |\mathbf{r}_1| = m_2 r/(m_1 + m_2)$, where $r = |\mathbf{r}|$. As $m_1\ddot{\mathbf{r}}_1 = \mathbf{F}$ and $m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}$, where $\mathbf{F} = -Gm_1m_2\mathbf{r}/r^3$, the equation for r is

$$\ddot{r} - \frac{h^2}{r^3} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{-Gm_1m_2}{r^2} = \frac{-G(m_1 + m_2)}{r^2} \qquad \text{where} \quad h = r^2\dot{\theta} \qquad \text{is constant.}$$

Comparing this with the usual orbit equation we concluded in lectures that the path of each particle is as if the other were stationary with a mass equal to the total mass. Alternatively, we can rewrite this equation in terms of r_1, in the form

$$\ddot{r}_1\frac{m_1 + m_2}{m_2} - \frac{h^2 m_2^3}{(m_1 + m_2)^3 r_1^3} = -\frac{G(m_1 + m_2)m_2^2}{(m_1 + m_2)^2 r_1^2}\ .$$

Rearranging, we obtain the standard orbit equation

$$\ddot{r}_1 - \frac{h_1^2}{r_1^3} = -\frac{GM_2}{r_1^2} \qquad \text{where} \quad h_1 = \frac{hm_2^2}{(m_1 + m_2)^2} \qquad \text{and} \quad M_2 = \frac{m_2^3}{(m_1 + m_2)^2}\ ,$$

as required. By symmetry, a similar formula holds for r_2. Each mass orbits about the centre of mass as if a mass M_2 were there, a situation I find much easier to visualise, even though the apparent mass M_2 is different for each mass.

Of course, if one of the masses is large, say $m_2 \gg m_1$, then the equivalent mass $M_2 \simeq m_2$. Furthermore, the centre of mass is then close to the centre of the more massive body. It is a very good approximation to treat the sun as stationary and us rotating around it.

Note, incidentally, that Kepler's 3rd law, stating that the constant value of a^3/τ^2 is the same for all planets is not exactly correct, as it depends on $(m_1 + m_2)$ where m_1 is the mass of the planet in question and m_1 is the mass of the sun. Again, this is a better approximation than most.

3. For a circular orbit, $\dot{r} = 0$ and $\ddot{r} = 0$. A radial force balance gives

$$GMm/r^2 = mr\dot{\theta}^2 \qquad \text{or} \quad r^3\dot{\theta}^2 = GM,$$

where M is the mass of the earth. We also need the tangential force, giving $r^2\dot{\theta}$ is constant, so that $\dot{\theta}$ is constant. The period of revolution is therefore $\tau = 2\pi/\dot{\theta}$, so that $4\pi^2 r^3 = \tau^2 GM$, which is Kepler's 3rd law, as for a circle the 'semi-major axis' is the radius, r.

For the orbit to be geo-stationary, it is necessary for the rotation rate of the satellite to be the same as that of the earth, or $\tau = 1$ day $= 24 \times 3600$ seconds. Using the values given in lectures, the orbital radius in metres is therefore given by

$$r^3 = \frac{(24 \times 3600)^2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{4\pi^2} \qquad \text{or} \quad r = 42.2 \times 10^6$$

This is about six or seven times the radius of the earth, $R_e = 6.37 \times 10^6$.

4. In a circular orbit, the velocity is $r\dot{\theta}$ in the θ-direction. Using Newton's second law in that direction gives

$$-mkr\dot{\theta} = m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \qquad \text{or} \quad \dot{h} = -kh$$

where $h = r^2\dot{\theta}$. Thus $h = h_0 e^{-kt}$ where $h = h_0$ at $t = 0$.

The radial component gives $r^3\dot{\theta}^2 = GM$, as in Q4. Thus $GM = (r^2\dot{\theta})^2/r = h^2/r$ so that $r = (h_0^2/GM)e^{-2kt}$, giving a fractional decay rate of $2k$. The kinetic energy,

$$K = \tfrac{1}{2}mr^2\dot{\theta}^2 = \tfrac{1}{2}\frac{mh^2}{r^2} = \tfrac{1}{2}\frac{mh_0^2(GM)^2}{h_0^4}e^{2kt} \ ,$$

so that the kinetic energy grows as the satellite falls. The loss in potential energy compensates both for the frictional disssipation and this increase in kinetic energy.

5. This question is much more conceptual. If we only consider the earth and the sun there is no direct way of telling whether the attractive force is gravitational or electrostatic. However, note that the electric charge on the earth would have to be of opposite sign to that of the sun. As the other planets also rotate around the sun, they would have to have the same sign of charge as the earth to be attracted to the sun. Therefore, the forces between the planets would be repulsive. In fact, observations of the orbits of planets show that these planetary interactions are attractive, and proportional to the mass of each body involved in the interaction. Indeed, some of the outer planets were discovered because of their gravitational pull on the known planets.

The orbit of the moon would also be hard to explain. It would be possible for it to have opposite signed charge from the earth, and to revolve about it while being repelled by the sun. But once more, the observations show that the moon is attracted by both earth and sun.