M1A1: Solutions to Problem Sheet 4

1. $\omega - k_1 = \omega - k + \sqrt{k^2-\omega^2} = \sqrt{k-\omega}\left(\sqrt{k+\omega} - \sqrt{k-\omega}\right) > 0$ as $k > \omega > 0$. Thus the decay rate k_1 of solutions for $k > \omega$ (overdamping) is less than for critical damping when the decay rate is ω.

2. For $0 < t < T$, $\ddot{x} + \omega^2 x = F_0$ has the general solution $x = A\cos\omega t + B\sin\omega t + F_0/(m\omega^2)$, for constants A and B. Imposing $x(0) = 0 = \dot{x}$, we have $x = F_0/(m\omega^2)(1-\cos\omega t)$.
For $t > T$, we have $x = C\cos(\omega t + \phi)$ for constants C and ϕ. Requiring **continuity** of x and $\dot{x}$ at $t = T$, we have

$$C\cos(\omega T + \phi) = \frac{F_0}{m\omega^2}(1-\cos\omega T) \qquad \text{and} \qquad -C\omega\sin(\omega T+\phi) = \frac{F_0}{m\omega}\sin\omega T\ .$$

Squaring and adding to eliminate ϕ,

$$C^2 = \left(\frac{F_0}{m\omega^2}\right)^2\left[(1-\cos\omega T)^2 + \sin^2\omega T\right] = \left(\frac{F_0}{m\omega^2}\right)^2(2-2\cos\omega T)$$

whence the oscillation amplitude $C = 2F_0\sin(\frac{1}{2}\omega T)/(m\omega^2)$, as required.

3. The total work, W, done by the force $F(t)$ is

$$W = \int_0^\infty F(t)\dot{x}\,dt = \int_0^T F_0\frac{F_0}{m\omega^2}\omega\sin\omega t\,dt = \frac{F_0^2}{m\omega^2}(1-\cos\omega T)\ .$$

The kinetic energy for $t > T$ is $K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(-\omega C\sin(\omega t+\phi))^2$. The potential (elastic) energy of the oscillator is $V = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2C^2\cos^2(\omega t+\phi)$. The total energy is therefore

$$K + V = \tfrac{1}{2}m\omega^2C^2 = \frac{2F_0^2}{m\omega^2}\sin^2\tfrac{1}{2}\omega T = W\ .$$

Thus as expected, the energy eventually acquired by the oscillator is equal to the work done by the driving force.

4. In equilibrium, the forces acting on the cup are gravity and tension. The tension is equal to λ times the extension of the elastic. Therefore $mg = \lambda(y_0 - x_0 - L)$ and $x_0 = y_0 - L - mg/\lambda$. When the system moves, the cup has a vertical acceleration $\ddot{x}$, so that

$$m\ddot{x} = -mg + \lambda(y - x - L) - 2mk\dot{x} = -mg + \lambda(y_0 + a\cos\omega_0 t - x - L) - 2mk\dot{x}.$$

Writing $X(t) = x(t) - x_0$, and using the equilibrium relation, we have

$$\ddot{X} + 2k\dot{X} + \omega^2X = F_0\cos\omega_0 t \qquad \text{where} \quad \omega^2 = \lambda/m \qquad \text{and} \quad F_0 = a\lambda/m\ .$$

If $k = 0$, the resonant frequency is $\omega_0 = \omega = \sqrt{\lambda/m}$. If $k = 0$ and $\omega = \omega_0$, we have

$$\ddot{X} + \omega^2X = F_0\cos\omega t \qquad \text{with} \quad X(0) = 0 = \dot{X}(0)\ .$$

This problem has the (resonant) solution $X = (F_0/2\omega)t\sin\omega t + A\cos(\omega t+\phi)$ and the boundary conditions imply $A = 0$. Thus the solution for x is $x = x_0 + (F_0/2\omega)t\sin\omega t$.

5. Resolving vertically, we have $N = mg - mh\dot{\theta}^2\cos\theta - mh\ddot{\theta}\sin\theta$, using the polar form of the acceleration. Taking moments, or using $\mathbf{F} = m\mathbf{a}$ perpendicular to the crutch, gives $mg\sin\theta = mh\ddot{\theta}$. Combining these, $N = m(g - g\sin^2\theta - h\dot{\theta}^2\cos\theta)$. We know (see skiier problem last sheet) that $N > 0$, so when $\theta = 0$ we must have $h\dot{\theta}^2 < g$ or $h\dot{\theta} < \sqrt{gh} \simeq 3.13m/s$. So if he manges a velocity of 3.5m/s he will leave the ground (again). This is why there is a maximum walking speed; after a while we start running.