M1A1: Solutions to Problem Sheet 3, 1-D Motion

1. Taking account of the sign of the velocity $v(x)$, the equation of motion is

$$\frac{dv}{dt} \equiv \frac{dv}{dx}\frac{dx}{dt} \equiv v\frac{dv}{dx} = \begin{cases} -g - \mu v^2 & \text{for} \quad v > 0 \\ -g + \mu v^2 & \text{for} \quad v < 0\ . \end{cases}$$

Initially $x = 0$ and $v = u > 0$. Integrating until $x = h$ when $v = 0$,

$$\int_u^0 \frac{v dv}{g + \mu v^2} = -\int_0^h dx \qquad \text{or} \qquad \left[\frac{1}{2\mu}\ln(g + \mu v^2)\right]_u^0 = -h\ .$$

Subsequently, $v < 0$ as the particle comes down. Its velocity $v(x)$ from then on is given by

$$\int_0^v \frac{-v dv}{g - \mu v^2} = \int_h^x dx \qquad \text{or} \qquad \left[\frac{1}{2\mu}\ln(g - \mu v^2)\right]_0^v = (x - h)\ .$$

Eliminating h (the maximum height reached) from these equations, we have

$$\left(1 - \frac{\mu v^2}{g}\right)\left(1 + \frac{\mu u^2}{g}\right) = e^{2\mu x}\ .$$

Thus when $x = 0$, we have $v^2 = gu^2/(g + \mu u^2)$ as required. The terminal velocity is attained as $x \to -\infty$, so that $v^2 \to g/\mu$, as expected from the ODE.

2. For an equilibrium, we require $V'(x) = 0$, or $-2a^2/x^3 + a/x^2 = 0$ so that $x = 2a$. Now $V''(x) = V_0/x^2(6a^2/x^2 - 2a/x)$, so that $V''(2a) = V_0/(8a^2) > 0$, so the equilibirum is stable. From lectures, the frequency ω of small oscillations about $x = 2a$ is given by

$$\omega^2 = \frac{V''(2a)}{m} = \frac{V_0}{(8ma^2)}.$$

At $x = 2a$, the potential takes the value $V(2a) = -\frac{1}{4}V_0$. As $x \to \infty$, $V \to 0$. To escape to infinity the particle thereore needs a kinetic energy $\frac{1}{2}mv^2 = \frac{1}{4}V_0$ so that the escape velocity is $v = \sqrt{V_0/(2m)}$.

3. The given function $V(x)$ is continuous at $x = 0$, but $V'(x)$ is discontinuous there. Clearly $V''(0)$ is not finite. For $x > 0$, the equation of motion is $\ddot{x} = -f$, or

$$x = -\tfrac{1}{2}ft^2 + ut \qquad \text{for} \quad 0 < t < 2u/f$$

using the given boundary conditions. Thus, $x = 0$ at $t = 0$ and $t = 2u/f$. When $t = 2u/f$, the particle has returned to $x = 0$ with speed $\dot{x} = -u$. For $x < 0$, the equation of motion is $\ddot{x} = -\omega^2 x$, which we recognise as SHM. The solution is

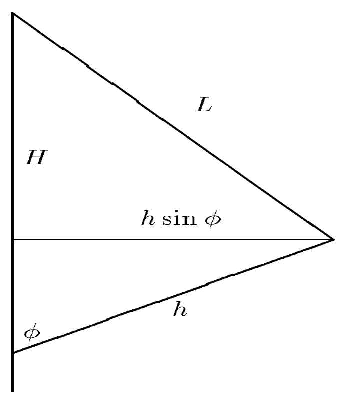
$$x = -\frac{u}{\omega}\sin\left[\omega\left(t - \frac{2u}{f}\right)\right] \qquad \text{for} \quad \frac{2u}{f} < t < \frac{2u}{f} + \frac{\pi}{\omega}.$$

The particle returns to $x = 0$ after a time π/ω, when its speed is u, again. The total time for an oscillation is therefore $T = \pi/\omega + 2u/f$.

4. The harness is a horizontal distance $h \sin\phi$ from the vertical cliff. Using Pythagoras' theorem, this is a vertical height

$$H = \sqrt{L^2 - h^2 \sin^2 \phi}$$

below the top of the cliff. A suitable potential energy function is therefore $V = -mgH$, as required. Equilibria occur when $dV/d\phi = 0$ which requires $\sin\phi\cos\phi = 0$. Possible equilibria are $\phi = 0$ (feet below harness, adjacent to cliff), $\phi = \pi$ (feet above harness, adjacent to cliff) and $\phi = \frac{1}{2}\pi$ (body perpendicular to cliff). For $\phi = 0, \pi$, we find that $V''(\phi) > 0$, so that these equilibria are stable, whereas for $\phi = \frac{1}{2}\pi$, $V''(\phi) < 0$ and the abseiler's intended position is unstable. Some friction with the cliff is necessary!



5. If $y = f(x)$ then $\dot{y} = f'(x)\dot{x}$ and $\ddot{y} = f'(x)\ddot{x} + f''(x)\dot{x}^2$.

Resolving forces vertically at the top of the hill at $x = a$, we have $N - mg = m\ddot{y}$. As $f'(a) = 0$ and $f''(a) = -c < 0$ (maximum height), this means $N = m(g - c\dot{x}^2)$ at this point.

Energy conservation requires that $mgy + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) =$ constant. At $x = a$, we know $\dot{y} = 0$ (provided he remains on the curve $y = f(x)$), so that at that point

$$mgh + \tfrac{1}{2}m\dot{x}^2 = mgH, \qquad \text{or} \quad N = mg[1 - 2c(H - h)]$$

as required. Clearly, if c is large enough, this expression will go negative, given that $H > h$. As normal reactions between surfaces only act in the direction to separate the two surfaces, this is **inconsistent**, and one of our assumptions is incorrect. A little thought reveals that our assumption that the skiier remains on the ground is incorrect, and in fact he will take off into the air (this is how ski-jumps work). Let's hope he lands safely.