M1A1: Solutions to Problem Sheet 2, Statics

1. Let a horizontal force T be applied, and let the block have mass M. Suppose μ is large enough for slipping not to occur. Then when the block is just about to topple over edge E, a vertical reaction N and a frictional force F will act at the edge E. Then

$$N = mg, \qquad F = T \qquad \text{and} \quad aMg = hT \qquad \text{so that} \quad \mu \geqslant \frac{F}{N} = \frac{T}{Mg} = \frac{a}{h}\ .$$

Thus if $\mu > a/h$, the block topples before it slips. Conversely, if $\mu < a/h$, equilibrium will break down before the block begins to topple, so that slipping must occur.

2. The condition for equilibrium does not depend on the weight of the ladder, merely on the position of its centre of mass. The ladder and person can be treated as a single system whose centre of mass moves up the ladder. If friction is strong enough to hold the ladder up when its centre of mass is at the midpoint, it will also suffice to hold the ladder/person while their centre of mass is lower than the midpoint. This is the case until the person is half way up, so it should always be safe to climb halfway (slowly).

3. Suppose N forces $\mathbf{F}_i$ act at positions $\mathbf{r}_i$. Then

$$\sum_{i=1}^{N} \mathbf{F}_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \mathbf{r}_i \wedge \mathbf{F}_i = 0\ .$$

The total moment about a point with position vector $\mathbf{R}$ is then

$$\sum_{i=1}^{N} (\mathbf{r}_i - \mathbf{R}) \wedge \mathbf{F}_i = \sum_{i=1}^{N} \mathbf{r}_i \wedge \mathbf{F}_i - \mathbf{R} \wedge \sum_{i=1}^{N} \mathbf{F}_i = 0 \qquad \text{as required}$$

4. For equilibrium, we have $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_0 = 0$, so that $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_0$ are coplanar, and

$$\mathbf{r}_1 \wedge \mathbf{F}_1 + \mathbf{r}_2 \wedge \mathbf{F}_2 = 0\ .$$

Taking the scalar product of this last equation with $\mathbf{F}_2$ implies $\mathbf{F}_2 \cdot (\mathbf{r}_1 \wedge \mathbf{F}_1) = 0$, so that the vectors $\mathbf{r}_1$, $\mathbf{F}_2$ and $\mathbf{F}_1$ must be coplanar. Similarly, taking the scalar product with $\mathbf{F}_1$, we have $\mathbf{F}_1 \cdot (\mathbf{r}_2 \wedge \mathbf{F}_2) = 0$ and $\mathbf{r}_2$, $\mathbf{F}_1$ and $\mathbf{F}_2$ are coplanar. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are not parallel, $\mathbf{F}_1$, $\mathbf{F}_2$ $\mathbf{r}_1$ and $\mathbf{r}_2$ must therefore be coplanar, and $\mathbf{F}_0$ also lies in this plane.
As $\mathbf{F}_1$ and $\mathbf{F}_2$ are not parallel, their lines of action must meet at a point P. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ have zero moment about this point. Therefore $\mathbf{F}_0$ must also have zero moment about P, so that its line of action must also pass through P.

5. The forces acting on the abseiler are the tension in the rope, a horizontal reaction off the ice wall, and a vertical weight. By Q2, the rope must pass through a point vertically above the centre of mass at the same height as his feet. If $h > d$, this requires his feet to be higher than his head.

6. The sliding block problem:

(1) The argument calculates the acceleration of P_1 relative to the pulley at the block corner. However, this point is itself accelerating. A frame with the pulley at rest would not be an inertial frame.

(2) f should balance the rate of change of the total horizontal momentum (see below.) There is an extra horizontal force acting on P_1 and the block due to the string pressing on the pulley.

Particle P_1 has a vertical acceleration $\ddot{y}$, so that

$$T - mg = m\ddot{y} \tag{1}$$

Particle P_2 has position vector $(x - L + a - y,\ a,\ 0)$. Its horizontal acceleration is therefore $\ddot{x} - \ddot{y}$, so that

$$T = m(\ddot{x} - \ddot{y}) \tag{2}$$

The centre of mass of the uniform block is at $(x - \frac{1}{2}a,\ \frac{1}{2}a,\ 0)$, so that its acceleration is $(\ddot{x},\ 0,\ 0)$ The horizontal forces acting on the block include a normal reaction from P_1 and a force from the string on the pulley. These can be calculated respectively by considering the horizontal forces acting on P_1 $(= m\ddot{x})$ and on the string (T).

Simplest is to consider the rate of change of horizontal momentum of the whole system, giving in either case,

$$f = \frac{d}{dt}\left[M\dot{x} + m\dot{x} + m(\dot{x} - \dot{y})\right] = (M + 2m)\ddot{x} - m\ddot{y} \tag{3}$$

Eliminating T from (1) and (2) gives

$$mg = m(\ddot{x} - 2\ddot{y}) \qquad \text{or} \quad g = \ddot{x} - 2\ddot{y} \tag{4}$$

Solving (3) and (4) gives

$$\ddot{x} = \frac{2f - mg}{3m + 2M}$$

and

$$\ddot{y} = \frac{f - 2mg - Mg}{3m + 2M}$$

as required.