M1A1: Solutions to Problem Sheet 1, Kinematics and Vectors

1. As the position vector $\mathbf{r}$ has dimensions of length, b must be a length and Ω a frequency with dimensions (time)$^{-1}$. Differentiating,

$$\mathbf{v} = \dot{\mathbf{r}} = b\Omega(-\sin\Omega t,\ \cos\Omega t,\ 0) \qquad \mathbf{a} = \dot{\mathbf{v}} = b\Omega^2(-\cos\Omega t,\ -\sin\Omega t,\ 0)$$

Thus $\mathbf{r}\cdot\mathbf{r} = b^2(\cos^2\Omega t + \sin^2\Omega t) = b^2$, $\mathbf{r}\cdot\mathbf{v} = 0$ and $\mathbf{r}\cdot\mathbf{a} = -b^2\Omega^2$.

2. Relative to the river bank, the woman's velocity is the sum of her velocity relative to the water and the water's velocity relative to the bank, or $(V_0 y(d-y)/d^2,\ u_0,\ 0)$. Thus if her position is $(x,\ y,\ 0)$

$$\dot{x} = V_0 y(d-y)/d^2\ , \qquad \dot{y} = u_0\ , \qquad \dot{z} = 0\ .$$

Integrating, $z = 0$, $y = u_0 t$ so that $\dot{x} = V_0 u_0 t/d - V_0 u_0^2 t^2/d^2$ or

$$x = \frac{V_0 u_0}{2d}t^2 - \frac{V_0 u_0^2}{3d^2}t^3.$$

She reaches the other bank when $y = d$, so that $t = d/u_0$ and $x = V_0 d/(6u_0)$.

3. Integrating twice, using the initial conditions,

$$\mathbf{r}_1 = \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{u}_1 t\ , \qquad \mathbf{r}_2 = \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{r}_0\ .$$

Thus the separation vector is $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{u}_1 t - \mathbf{r}_0$. Now

$$\begin{aligned}|\mathbf{r}_1 - \mathbf{r}_2|^2 &= (\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2) = (\mathbf{u}_1 t - \mathbf{r}_0)\cdot(\mathbf{u}_1 t - \mathbf{r}_0)\\ &= |\mathbf{u}_1|^2 t^2 - 2\mathbf{u}_1\cdot\mathbf{r}_0 t + |\mathbf{r}_0|^2 = t^2|\mathbf{u}_1|^2 - 2t|\mathbf{u}_1||\mathbf{r}_0|\cos\theta + |\mathbf{r}_0|^2\\ &= |\mathbf{u}_1|^2\left(t - \frac{|\mathbf{r}_0|\cos\theta}{|\mathbf{u}_1|}\right)^2 + |\mathbf{r}_0|^2 - \cos^2\theta|\mathbf{r}_0|^2\end{aligned}$$

completing the square. Clearly this expression is minimum when $t = |\mathbf{r}_0|\cos\theta/|\mathbf{u}_1|$. Thus the minimum value of $|\mathbf{r}_1 - \mathbf{r}_2|$ is $|\mathbf{r}_0|\sin\theta$.

4. The centre of the wheel has velocity $(v,\ 0, 0)$, so a general point has velocity $\mathbf{v}$

$$\mathbf{v} = (v,\ 0, 0) + (0,\ \omega,\ 0)\wedge(x,\ y,\ z) = (v + \omega z,\ 0,\ -\omega x).$$

The contact point with the ground is $(0,\ 0,\ -R)$, at which point the velocity is $\mathbf{v} = (v - \omega R,\ 0,\ 0)$. For no slipping, we must have $v = \omega R$.
The top half of a wheel ($z > 0$) thus moves faster than the bottom half. As a result, the spokes are visible near the bottom, but often move too fast for the eye near the top.

5. From Newton's second and third laws we have $m_1\ddot{\mathbf{r}}_1 = \mathbf{F}$ and $m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}$. Now

$$\begin{aligned}\frac{d\mathbf{H}}{dt} &= m_1(\mathbf{r}_1\wedge\ddot{\mathbf{r}}_1 + \dot{\mathbf{r}}_1\wedge\dot{\mathbf{r}}_1) + m_2(\mathbf{r}_2\wedge\ddot{\mathbf{r}}_2 + \dot{\mathbf{r}}_2\wedge\dot{\mathbf{r}}_2)\\ &= m_1\mathbf{r}_1\wedge\ddot{\mathbf{r}}_1 + m_2\mathbf{r}_2\wedge\ddot{\mathbf{r}}_2 = \mathbf{r}_1\wedge\mathbf{F} + \mathbf{r}_2\wedge(-\mathbf{F}) = (\mathbf{r}_1 - \mathbf{r}_2)\wedge\mathbf{F}\ .\end{aligned}$$

Thus if $\mathbf{F}$ is parallel to $(\mathbf{r}_1 - \mathbf{r}_2)$ we have $\dot{\mathbf{H}} = 0$ or $\mathbf{H}$ is constant. This important result is known as the **Conservation of Angular Momentum.**