We have seen how useful potentials are for understanding motion where energy is conserved, that is, when the forces acting are **conservative**. We have dealt with 1-D potentials $U(x)$, and spherically symmetric potentials, $U(r)$. This last, is really a three-dimensional potential, which happens to be spherically symmetric; more generally, we could think about a potential $U(x, y, z)$.

Suppose at each point in 3-D space, we can assign a potential energy $U(x, y, z)$. Energy conservation for a particle of mass m with velocity $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ then implies

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z) = E, \text{ a constant.}$$

We now differentiate this equation with respect to time. We recall the chain rule for partial derivatives from M1M2, to give

$$m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) + \dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y} + \dot{z}\frac{\partial U}{\partial z} = 0 \ ,$$

or in terms of vectors,

$$m\mathbf{v} \cdot \ddot{\mathbf{r}} = -\mathbf{v} \cdot \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \ .$$

Comparing with Newton's second law $\mathbf{F} = m\ddot{\mathbf{r}}$, we see this strongly suggests the relation

$$\mathbf{F} = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \equiv -\nabla U \ . \tag{5.5}$$

We have thus defined a very important symbol, ∇, pronounced "del" or "grad" It operates on a scalar function of position, and gives a vector of partial derivatives.

Note: (1) The function $U(x, y, z)$ is often called a **scalar field.** This just means it is defined throughout a region of 3-D space. Correspondingly, $\mathbf{F}$ is called a **vector field**. Next year's course M2M1 is all about vector fields.

(2) Not every force field $\mathbf{F}$ can be written in the form (5.5). But then, not every force field is conservative. In general, a **force F is conservative if and only if it can be written in the form**

$$\mathbf{F} = -\nabla U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \ .$$

(3) For a particle to be in equilibrium at (x_0, y_0, z_0) it is necessary for $\mathbf{F} = 0$ or $\nabla U = 0$ there. This is a **stable equilibrium** if and only if U has a **minimum** there. Think of U as representing a reservoir of available energy. At a minimum no energy can be extracted from the potential field to be converted into kinetic energy, and the particle is trapped, just as in one-dimension.

(4) Potentials can be added: if two forces $\mathbf{F}_1 = -\nabla U_1$ and $\mathbf{F}_2 = -\nabla U_2$ act, then the net force $\mathbf{F}_1 + \mathbf{F}_2 = -\nabla(U_1 + U_2)$, so that the total potential $U = U_1 + U_2$.

This allows us to calculate the gravitational potential of any mass distribution, as follows:

Gravitational Potential for Arbitrary Mass Distributions:

The potential for a mass m at position $\mathbf{r}$ due to a mass m_0 at the origin is $-Gmm_0/r$. Thus the potential due to masses m_1 and m_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ is

$$U = -\frac{Gmm_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{Gmm_2}{|\mathbf{r} - \mathbf{r}_2|} \tag{5.6}$$

This can easily be generalised to N particles. More usefully, we can define the **mass density** $\rho(x, y, z)$, which we also write as $\rho(\mathbf{r})$. This is a scalar field, defined so that the mass within any small volume δD surrounding the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ is $\rho(x_0, y_0, z_0)\delta D$. The potential at point $\mathbf{r}$ due to this small volume is thus $U = -Gm\rho(\mathbf{r}_0)\delta D/|\mathbf{r} - \mathbf{r}_0|$. Combining the potentials from these small volumes we obtain the total potential due to all mass (in the universe!)

$$U(x, y, z) = -Gm \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\rho(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|}\, dx_0\, dy_0\, dz_0 \ . \tag{5.7}$$

Consider a mass distribution with a spherically symmetric density, $\rho(r)$ [This is a special case of $\rho(\mathbf{r})$]. In terms of spherical polar coordinates (r_0, θ_0, ϕ_0), with the axis pointing along $\mathbf{r}$, $|\mathbf{r} - \mathbf{r}_0|^2 = r^2 + r_0^2 - 2rr_0 \cos\theta_0$, so that the expression in (5.7) can be written

$$U(r) = -Gm \int_0^{\infty} \int_0^{\pi} \frac{\rho(r_0) 2\pi r_0 \sin\theta_0}{(r^2 + r_0^2 - 2rr_0 \cos\theta_0)^{1/2}}\, r_0\, d\theta_0\, dr_0 \ .$$

The θ_0-integration can be done giving

$$U(r) = -\frac{2\pi Gm}{r} \int_0^{\infty} r_0 \rho(r_0) \left[(r^2 + r_0^2 - 2rr_0 \cos\theta_0)^{1/2} \right]_0^{\pi} dr_0$$

Substituting the limits the square roots disappear. However we must be careful, as $\sqrt{(r - r_0)^2} = r - r_0$ if $r > r_0$, but $r_0 - r$ if $r < r_0$. Splitting the integral into two parts, we find

$$U(r) = -4\pi Gm \left[\int_0^r \rho(r_0) \frac{r_0^2}{r}\, dr_0 + \int_r^{\infty} r_0 \rho(r_0)\, dr_0 \right] \ . \tag{5.8}$$

For example, consider the potential outside a uniform sphere of radius a, so that $\rho(r_0) = \rho_0$ for $r_0 < a$ and $\rho = 0$ for $r_0 > a$. In the second integral, as $r_0 > r > a$, ρ is zero and we find

$$U(r) = -\frac{Gm(\frac{4}{3}\pi a^3 \rho_0)}{r} = -\frac{GmM_0}{r} \ . \tag{5.9}$$

where M_0 is the total mass of the sphere. The gravitational field outside the sphere is the same as if all the mass were concentrated at the centre of the sphere.

If we differentiate (5.8), we can find the radial gravitational force,

$$F(r) = -\frac{dU}{dr} = -\frac{Gm}{r^2} \int_0^r \rho(r_0) 4\pi r_0^2 dr_0 = -\frac{GmM(r)}{r^2} \ , \tag{5.10}$$

where $M(r)$ is the total mass inside the sphere of radius r. Interestingly, all mass a distance greater than r from the centre of spherical symmetry does not contribute to the force, while all mass nearer the centre behaves as a point particle at the centre.

Don't miss next year's instalment, M2M1 Vector Fields!