In one-dimensional motion, a particle of mass m with position vector $\mathbf{r} = (x(t),\, 0,\, 0)$ under the influence of a force $\mathbf{F} = (F,\, 0,\, 0)$ where $F = F(x,\, t,\, v)$, obeys the equation

$$F = m\ddot{x} \equiv \dot{v} \qquad \text{where} \quad v = \dot{x}\ .$$

If F only depends on t then this equation may be integrated twice to find $x(t)$ in terms of the initial values $x(0)$ and $\dot{x}(0)$. Likewise, if F depends only on v, the equation is separable in v and can in principle be integrated once to find $v(t)$ and again to find $x(t)$. A third important case is if F depends only on x, so that we write $F = F(x)$. Such forces are **conservative**, in that we can define a **potential energy** or **potential** $U(x)$ by

$$F = -\frac{dU}{dx} \qquad \text{or} \quad U(x) = -\int F(x)\,dx \equiv -\int F(x)v\,dt\ . \tag{2.4}$$

In the resulting motion, the sum of the kinetic and potential energy, E, is constant. For

$$E = \tfrac{1}{2}mv^2 + U(x) \quad \Longrightarrow \quad \frac{dE}{dt} = mv\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt} = v\left(m\frac{dv}{dt} - F\right) = 0\ . \tag{2.5}$$

This is the principle of **Conservation of Energy**. For example, under constant gravity, $F = -mg$ if the x-axis is directed upwards, so that $U = mgx$.

Another example: When an elastic string or spring is extended from its natural length a to $a + x$ then it exerts a force $F = -kx$. Thus when a particle is attached to a fixed point by such a spring, the associated potential energy (or **elastic energy**) is

$$U(x) = \int kx\,dx = \tfrac{1}{2}kx^2 \qquad \text{so that} \quad \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E \quad \text{(constant)} \tag{2.6}$$

in the resulting motion.

We can in principle integrate the energy equation (2.5), so that

$$\frac{dx}{dt} = \frac{2}{m}\sqrt{E - U(x)} \qquad \text{or} \quad \frac{2t}{m} = \int \frac{dx}{[E - U(x)]^{1/2}}. \tag{2.7}$$

However, this integral is frequently difficult to evaluate, and we can learn a tremendous result about the ensuing motion just by looking at $U(x)$.

Firstly, we note that the initial conditions fix the constant energy E. Thenafter, as the kinetic energy $K = \frac{1}{2}mv^2 \geqslant 0$, the speed $|v|$ can only increase if the potential U decreases. In particular, values of x for which $U(x) > E$ can **never be attained.** The particle can therefore be trapped by the potential in a suitable x-interval. For an **equilibrium** to occur at some value of x, it is necessary both for the force to vanish, so that $F = -U'(x) = 0$, and also for the particle speed $v = 0$ there. For example, consider the particle attached to the spring as in (2.5). We sketch the curves $y = U(x) = \frac{1}{2}kx^2$ and the horizontal line $y = E$. Only points where $E > U$ are accessible, so that the particle is trapped between $x = \pm\sqrt{2E/k}$. Its speed is only zero at these end points and as it cannot pass them it must change sign when it reaches them. As $U' = 0$ only at $x = 0$, there is no equilibrium unless $E = 0$. We conclude that the particle oscillates endlessly between $x = \sqrt{2E/k}$ and $x = -\sqrt{2E/k}$. We learn this from general arguments rather than by solving the equation. However, for this simple probem, we can verify our conclusions by finding $x(t)$ directly. Newton's second law states

$$m\ddot{x} + kx = 0 \qquad \text{which we recognise from M1M2 as having the solution}$$

$$x = A\cos\omega t + B\sin\omega t \equiv C\cos(\omega t + \phi) \qquad \text{where} \quad \omega^2 = k/m \ . \tag{2.8}$$

Here A and B or equivalently C and ϕ are arbitrary constants to be determined from the initial conditions. Clearly, x oscillates between $\pm C$. We can verify the equivalence of this solutions and the energy equation by noting $v = \dot{x} = -C\omega\sin(\omega t + \phi)$ so that

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mC^2\omega^2\sin^2(\omega t + \phi) + \tfrac{1}{2}kC^2\cos^2(\omega t + \phi) = \tfrac{1}{2}kC^2 \ .$$

This agrees with our earlier conclusions when we identify $E = \frac{1}{2}kC^2$. Usually, it is harder to solve the ODE, but the potential still tells us everything we need to know.

Example: A particle of mass m moves along the x-axis subject to the force $F(x) = -m\omega^2 x + \frac{3}{2}m\omega^2 x^2/a$ where ω and a are constants. Discuss the possible motion.

Stability of Equilibria: Frequency of Small Oscillations:

The conditions for $x = a$ to be an **equilibrium** are that

$$U'(a) = 0 \qquad \text{[No force]} \qquad \text{and} \quad U(a) = E \qquad \text{[No velocity]} \quad .$$

However, in practice it is vital to know whether the equilibrium is **stable**. The question is: if the particle is disturbed slightly from its equilibrium position, does it remain close to the equilibrium position, or does it wander a long way away as time increases? From consideration of the potential curve, it is easy to see that if $U(x)$ has a minimum at $x = a$, then if E is increased slightly from the value $U(a)$, the particle will be trapped between two values of x close to a. Correspondingly, if $U(a)$ is a local maximum, there will be no nearby values of x at which $U(x) = E$ so that the particle velocity can change sign. As a result, such equilibria are **unstable.** We now analyse the motion.

Suppose the particle is disturbed from $x = a$ to $x = a + \varepsilon\eta(t)$ where $0 < \varepsilon \ll 1$. Then assuming η remains small, we may expand $U(x)$ as a Taylor series:

$$\begin{aligned} U(x) = U(a + \varepsilon\eta) &= U(a) + \varepsilon\eta U'(a) + \tfrac{1}{2}\varepsilon^2\eta^2 U''(a) + O(\varepsilon^3) \\ &= U(a) + \tfrac{1}{2}\varepsilon^2\eta^2 U''(a) + O(\varepsilon^3) \end{aligned}$$

as $U'(a) = 0$ since $x = a$ is an equilibrium point. Thus the enrgy equation (2.4) becomes

$$\tfrac{1}{2}m\varepsilon^2\dot{\eta}^2 + \tfrac{1}{2}\varepsilon^2 U''(a)\eta^2 \simeq \text{constant}, \qquad \text{or differentiating,}$$

$$m\dot{\eta}\ddot{\eta} + U''(a)\eta\dot{\eta} = 0 \quad \Longrightarrow \quad \ddot{\eta} + (U''(a)/m)\eta = 0 \ .$$

Comparison with the equation (2.5) shows that provided $U''(a) > 0$, the article will oscillate with frequency ω given by

$$\omega^2 = U''(a)/m \ . \tag{2.9}$$

If however $U''(a) < 0$, so that $x = a$ is a **maximum** of potential, then instead of oscillating the ODE has exponentially growing solutions. In this case the equilibrium is **unstable.**