UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2001

This paper is also taken for the relevant examination for the Associateship

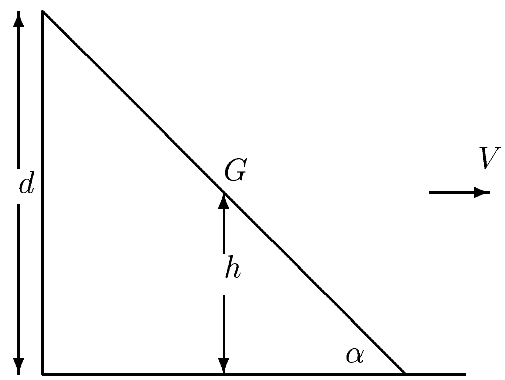
M1A1 Mechanics

DATE: Wednesday 23rd May 2001 TIME: 10 – 12

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. A particle has position vectors $\mathbf{R}$ and $\mathbf{r}$ in, respectively, an inertial frame and a frame with moving origin P. The point P has position vector $\mathbf{S}$ in the inertial frame. Show that Newton's second law applies in the noninertial frame, provided that a fictitious force, $-m\ddot{\mathbf{S}}$ is included.



A tube train travels with speed $V > 0$ along the x-axis. A passenger is leaning backwards against a smooth wall at an angle α as in the diagram, when the train begins to decelerate at a constant rate. The man's centre of mass G is a height h above the floor, while his back touches the wall at a height $d > h$. The coefficient of friction between his feet and the floor is μ.

Write down the force and moment equations assuming that the man remains in static equilibrium relative to the train.

Show that the condition that his feet do not slip from under him is

$$\mu g d \geq a(d-h) + gh\cot\alpha \qquad \text{where} \quad a = -\dot{V},$$

but that he will begin to topple forwards if $a > g\cot\alpha$.

What leaning angle gives an equilibrium for the greatest value of a/g?

Turn over...

2. A pendulum consists of a mass m suspended from a fixed point by a light inextensible string of length d. The string makes an angle x with the downwards vertical. When can the motion be described by the equation $\ddot{x} + \omega^2 x = 0$ for some constant ω?

A second identical pendulum feels slight vibrations due to the oscillation of the first, and an equal and opposite force is felt by the first pendulum. The governing equations are thus

$$\ddot{x} + \omega^2 x = -\varepsilon \omega^2 x \ , \qquad \ddot{y} + \omega^2 y = \varepsilon \omega^2 x \ ,$$

where $|\varepsilon| \ll 1$. At time $t = 0$

$$x = x_0, \qquad y = 0, \qquad \dot{x} = \dot{y} = 0 \ .$$

Find $x(t)$ and $y(t)$.

Show from your solution that provided $|\varepsilon \omega t| \ll 1$,

$$y \simeq \tfrac{1}{2} \varepsilon x_0 \omega t \sin \omega t \ .$$

3. A particle of mass m moves along the x-axis under the influence of a potential $V(x)$. What is the condition for $x = x_0$ to be an equilibrium point?

Writing $x = x_0 + \varepsilon f(t)$, where $0 < \varepsilon \ll 1$, show that the frequency, ω, of small oscillations about the equilibrium point $x = x_0$ is

$$\omega = \left(V''(x_0)/m\right)^{1/2} .$$

What happens if $V''(x_0) < 0$?

Suppose now that

$$V(x) = \frac{V_0}{a^4}\left(12x^2a^2 - 4ax^3 - 3x^4\right) ,$$

where V_0 and a are constant. Find the equilibrium points and discuss their stability.

At $t = 0$, $x = 0$ and $\dot{x} = u$. Fow what values of u does the particle

(a) oscillate about an equilibrium point

(b) escape to $+\infty$

(c) escape to $-\infty$?

4. A long way from the origin, O, a particle of mass m is moving along a straight line L with speed v. The shortest distance from O to L is b. What are the particle's kinetic energy and angular momentum about O?

As the particle approaches O, it is acted upon by a force $F(r) = km/r^2$ directed towards O, where k is constant and (r, θ) are polar coordinates. The line $\theta = 0$ is defined to be parallel to L. Write down the polar components of the equation of motion, and relate $r^2\dot{\theta}$ to the incoming motion.

After some time, the particle is again a long way from O, its direction of motion having been deflected through an angle ϕ. What is its final speed, and what is the net change in the component of momentum parallel to L?

Give a physical interpretation of the time integral

$$P = \int_{-\infty}^{\infty} F(r) \cos\theta \, dt$$

and hence or otherwise, show that the deflection angle ϕ is given by

$$\tan \tfrac{1}{2}\phi = \frac{k}{bv^2} .$$

5. With respect to an inertial frame, the x and y-axes are rotating with constant angular velocity $\underline{\omega} = (0, 0, \omega)$. A position vector in the plane $z = 0$ is written $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$, where the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ point along the rotating coordinate axes. Show that

$$\ddot{\mathbf{r}} = (\ddot{x} - 2\dot{y}\omega - x\omega^2)\hat{\mathbf{x}} + (\ddot{y} + 2\dot{x}\omega - y\omega^2)\hat{\mathbf{y}} \ ,$$

and identify the various terms in this expression.

A stone is dropped from the top of a tower of height h on the equator. The x-axis points East, and the y-axis points towards the centre of the (spherical) earth. Ignoring air resistance, variations in gravity and terms quadratic in ω, show that when the stone hits the ground

$$x = \left(\frac{8h^3\omega^2}{9g}\right)^{1/2} \ .$$