In 1666, while London was racked by the great fire, a much more significant event took place 100 miles North, as Isaac Newton propounded

Newton's Law of Gravitation: Two particles of masses m_0 and m_1 separated by a distance r are attracted towards each other by a force of magnitude

$$F = \frac{Gm_0 m_1}{r^2} \qquad \text{where } G \text{ is the } \textbf{gravitational constant.}$$

Of the various physical constants which define how the universe works, G is perhaps the hardest to measure accurately, but $G \simeq 6.67 \times 10^{-11}$ in standard units. (What are its physical dimensions?) The acceleration of particle m_1 due to the presence of m_0 is thus the same whatever its mass. There is actually a subtle question here, as one can distinguish two kinds of mass, "inertial mass" (as in $\mathbf{F} = m\mathbf{a}$) measuring how hard it is to cause a particle to start moving, and gravitational mass (how hard a particle pulls at another.) It seems to be a physical fact that these are the same.

The **gravitational potential**, V, due to a mass M is thus

$$V(r) = -\frac{GMm}{r} \qquad \text{so that} \quad V_{\text{eff}}(r) = \frac{mh^2}{2r^2} - \frac{GMm}{r}$$

where $h = r^2\dot{\theta}$ is the angular momentum per unit mass. The orbit of the particle m is given by the energy equation

$$\tfrac{1}{2}m\dot{r}^2 + V_{\text{eff}} = E \qquad \text{a constant.}$$

Drawing V_{eff} against r, we see that negative values of E correspond to oscillatory behaviour, whereas if $E > 0$, the mass m escapes to infinity.

Planetary orbits: Returning to the orbit equation (3.7), we see that a planet of mass m orbiting the sun of mass M feels a force

$$F(r) = -\frac{GMm}{r^2} \qquad \text{so that} \quad \frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} \equiv \frac{1}{l} \tag{3.9}$$

where l is constant. The particular integral for u is $u = 1/l$, so the general solution is

$$u = \frac{1}{l}\left[1 + e\cos(\theta + \phi)\right] \qquad \text{for constant } \phi \text{ and } e.$$

Without loss of generality, $e \geqslant 0$, while for any particular orbit we can choose the definition of θ in such a way that $\phi = 0$. This leaves us with the standard orbit equation

$$\frac{l}{r} = 1 + e\cos\theta\,. \tag{3.10}$$

This describes a curve in the (r, θ)-plane. What does it look like? First, note that r is a 2π-periodic function of θ, so that a bounded orbit must be closed. When $e = 0$, $r = l$ and we have a circle. If $0 < e < 1$, l/r oscillates between $(1-e)$ and $(1+e)$ so that the orbit is bounded. If $e \geqslant 1$, however, $r \to \infty$ as $\cos\theta \to -1/e$, and the planet escapes to infinity. We can rewrite (3.10) in Cartesian coordinates,

$$l = r + er\cos\theta = \sqrt{x^2+y^2} + ex \qquad \text{so that} \quad l - ex = \sqrt{x^2+y^2}\ .$$

Squaring, and rearranging, we have

$$x^2(1-e^2) + 2elx + y^2 = l^2 \qquad \text{or} \qquad \left(x + \frac{el}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{l^2}{(1-e^2)^2}\ .$$

We recognise this curve as a **conic**. When $e = 0$ it is a circle, when $0 < e < 1$ it is an ellipse, when $e = 1$ it is a parabola and when $e > 1$ it is a hyperbola. The parameter e is called the **eccentricity** of the orbit. All planets, asteroids, comets etc. have elliptical orbits, if we ignore the interactions between them. Comparing with the standard equation for an ellipse, $X^2/a^2 + Y^2/b^2 = 1$, we see that

$$a = \frac{l}{1-e^2}, \quad b = \frac{l}{\sqrt{1-e^2}}, \quad e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}, \quad l = \frac{b^2}{a}\ . \tag{3.11}$$

The centre of the ellipse is at $x = -ae$, so that the sun is at a **focus** of the ellipse.

Kepler's laws: Newton's theory was to a large extent deduced from observations by Tycho de Brahe interpreted by Johann Kepler, whose two laws from 1609 and a third from 1618, described how the planets move:

(1) The orbits are ellipses with the sun at one of the foci.

(2) The radius vector from the sun sweeps out equal areas in equal times.

(3) The square of the orbital period of each planet is proportional to the cube of the semi-major axis (a, above), and the constant of proportionality is the same for each planet.

We have verified that Newton's inverse-square-law for gravitation agrees with the first of Kepler's laws. The second holds for any **central force**, as it is essentially angular momentum conservation. The area $A(\theta)$ of a curve $r = r(\theta)$ can be written as $A = \frac{1}{2}\int_0^\theta r^2\,d\theta$ so that $dA/dt = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h$. Kepler's third law then follows from the first two. The orbital period (or 'year'), τ, is the time taken for a complete revolution, during which time an entire ellipse is swept out by the radius. Thus $\tau = 2A_e/h$, where $A_e = \pi ab$ is the area of an ellipse. Using (3.11) and (3.9), we have

$$\tau^2 = \frac{4\pi^2a^2b^2}{h^2} = \frac{4\pi^2a^2b^2}{GMl} = \frac{4\pi^2}{GM}a^3\ .$$

Some astronomical data: The sun's mass is 1.99×10^{30}kg, while the earth's is 5.98×10^{24}kg and the moon's 7.34×10^{22}kg. The eccentricity of the earth's orbit is small, $e = 0.017$. Most planets have approximately circular orbits, but for Pluto $e \simeq 0.25$ and Mercury $e \simeq 0.2$. As a result, in 1999, Pluto was nearer the sun than Neptune! Comets have highly elliptical orbits, and so take a long time to return; Halley's comet, for example has $e = 0.967$. The earth is 1.49×10^{11}m from the sun, and 3.84×10^{8}m from the moon. Its radius is 6.37×10^{6}m.