**Kelvin-Helmholtz and Rayleigh-Taylor Instabilities**

Consider two fluid regions, $y > 0$ and $y < 0$. In $y > 0$ let the fluid have uniform velocity $\mathbf{u} = (U_1, 0, 0)$ and constant density $\rho = \rho_1$, whereas in $y < 0$ let $\rho = \rho_2$ and $\mathbf{u} = (U_2, 0, 0)$. Gravity is assumed to act in the negative $y$-direction, $\mathbf{g} = (0, -g, 0)$. This configuration could represent wind blowing over a lake, or a model of separated flow over a step.

We will assume the flow is inviscid. It can therefore suffer the tangential velocity discontinuity at $y = 0$. The vortex sheet associated with this discontinuity would diffuse outwards if $\mu \neq 0$, but we will neglect this spreading and assume that the interface has the shape $y = \varepsilon h(x, t)$. As there is no vorticity elsewhere in the flow initially, we expect the flow to be irrotational, i.e.

$$\mathbf{u} = \nabla\phi \qquad \text{where} \quad \nabla^2\phi = 0. \tag{1.1}$$

Then we write

$$\phi = U_1 x + \varepsilon\phi_1 \qquad \text{in} \quad y > 0, \qquad \phi = U_2 x + \varepsilon\phi_2 \qquad \text{in} \quad y < 0\ . \tag{1.2}$$

We want $\phi_1 \to 0$ as $y \to \infty$ and $\phi_2 \to 0$ as $y \to -\infty$. The kinematic boundary condition takes the form on the interface

$$0 = \frac{D}{Dt}(y - \varepsilon h) = \frac{\partial\phi}{\partial y} - \varepsilon\frac{\partial h}{\partial t} - \varepsilon\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}\ . \tag{1.3}$$

or neglecting terms of $O(\varepsilon^2)$,

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial h}{\partial t} + U_1\frac{\partial h}{\partial x} \qquad \text{and} \qquad \frac{\partial\phi_2}{\partial y} = \frac{\partial h}{\partial t} + U_2\frac{\partial h}{\partial x}\ . \tag{1.4}$$

To leading order we can evaluate (1.4) on $y = 0$ rather than $y = \varepsilon h$. The other condition to apply is that the pressure must be continuous across the interface or if we ignore surface tension for the moment $p_1 = p_2$ on $y = \varepsilon h$. The time-dependent Bernoulli condition is from (0.14)

$$p + \rho\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + gy\right) = \text{constant},$$

so that to leading order on the interface $y = \varepsilon h$,

$$\rho_1\left(\frac{\partial\phi_1}{\partial t} + U_1\frac{\partial\phi_1}{\partial x} + gh\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + U_2\frac{\partial\phi_2}{\partial x} + gh\right)\ . \tag{1.5}$$

Once more, we can evaluate (1.5) on $y = 0$ rather than $y = \varepsilon h$.

As neither $x$ or $t$ appears in the coefficients of the problem, we can seek a solution proportional to $e^{ikx+st}$. Here $k > 0$ is a real **wave-number**, and $s$ is possibly complex. If we can find a value of $k$ for which the corresponding $s$ has a positive real part ($\Re e(s) > 0$), then the interface $y = 0$ is **unstable.** We write

$$h = h_0 e^{ikx+st}, \qquad \phi_1 = \Phi_1(y)e^{ikx+st}, \qquad \phi_2 = \Phi_2(y)e^{ikx+st}\ .$$

Then $\Phi_1(y)$ satisfies the ODE and boundary conditions

$$\Phi_1'' - k^2\Phi_1 = 0, \qquad \Phi_1 \to 0 \qquad \text{as} \quad y \to \infty \qquad \Phi_1'(0) = h_0(s + ikU_1) \tag{1.6}$$

from which it follows that

$$\Phi_1(y) = -\frac{h_0}{k}(s + ikU_1)e^{-ky} \qquad \text{and similarly} \quad \Phi_2(y) = \frac{h_0}{k}(s + ikU_2)e^{ky} \ . \tag{1.7}$$

The pressure constraint (1.4) requires that

$$\rho_1 \left[(s + ikU_1)\Phi_1(0) + gh_0\right] = \rho_2 \left[(s + ikU_2)\Phi_2(0) + gh_0\right] \tag{1.8}$$

or combining (1.8) and (1.7),

$$\rho_1 \left[gk - (s + ikU_1)^2\right] = \rho_2 \left[gk + (s + ikU_2)^2\right] \ . \tag{1.9}$$

This is a quadratic equation for $s(k)$, which is in general known as the **dispersion relation**. Solving this equation we obtain

$$s = -ik\,\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + kg\,\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right]^{1/2} \ . \tag{1.10}$$

For instability, $s$ must have a positive real part. It is clear that this will occur if and only if the quantity in square brackets is positive, that is if

**Instability if:** $$k^2\rho_1\rho_2(U_1 - U_2)^2 > kg(\rho_2^2 - \rho_1^2) \ . \tag{1.11}$$

**Instability of a vortex sheet:** If $U_1 \neq U_2$, we see from (1.11) that instability always occurs for large enough $k$, i.e. for short wavelengths. The smaller the wavelength the larger the growth rate. This is known as the Kelvin-Helmholtz instability.

**Water waves:** For example, consider air moving over a stationary lake, so that $U_2 = 0$, $\rho_2 \gg \rho_1$. Instability occurs if

$$k > \frac{g}{U_1^2}\frac{\rho_2}{\rho_1} \tag{1.12}$$

**Heavy fluid over light fluid:** If $\rho_1 > \rho_2$ we see that every value of $k$ leads to instability (recall $k > 0$). This is known as the Rayleigh-Taylor instability.

**Wave frequencies:** If $U_1 = U_2 = 0$ and $\rho_1 < \rho_2$ then the interface can support surface waves of frequency $\omega$ ($s = i\omega$) where

$$\omega = \sqrt{gk}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)^{1/2} \ . \tag{1.13}$$

The phase velocity $c = \omega/k$ for small water waves ($\rho_1 = 0$) is therefore $c = (g/k)^{1/2}$.

### Three dimensional disturbances:

So far we have only considered 2-D perturbations. If instead we allow the surface to take the form

$$y = \varepsilon h_0 e^{ikx+ilz+st} \qquad \text{where} \quad \kappa = (k^2 + l^2)^{1/2} \ , \tag{1.14}$$

then the analysis is very similar, with for example $\Phi_1 \propto e^{\kappa y}$. The instability condition (1.11) remains the same with the "$k$" on the RHS replaced by "$\kappa$." As $\kappa \geqslant k$, we can infer from this that if a mode with a particular $(k, l)$ is unstable so is the mode with $(k, 0)$, and indeed the growth rate is **larger** for the 2-D disturbance. We will meet this idea later – an example of **Squires' theorem.**

If a configuration is unstable, we have shown that the largest growth rates have large $k$, and are formally infinite. In practice, some physical effect we have neglected will become important. Two processes we might expect to limit the size of the wave-number and growth-rates are **viscosity** and **surface tension.**

### The effect of Surface Tension

The curvature of the surface (1.14) is

$$K = \nabla \cdot \widehat{\mathbf{n}} = \nabla \cdot [(-\varepsilon ikh_0,\, 1,\, -\varepsilon ilh_0)e^{ikx+ilz+st}] = \varepsilon\kappa^2 h_0\, e^{ikx+ilz+st} \ . \tag{1.15}$$

The normal stress condition $p_1 = p_2$ is now replaced by $p_1 = p_2 + \gamma K$. Making this modification, the dispersion relation (1.10) takes the form

$$s = -ik\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{\kappa g(\rho_1 - \rho_2) - \kappa^3\gamma}{\rho_1 + \rho_2}\right]^{1/2} \ . \tag{1.16}$$

The cubic term $\gamma\kappa^3$ will clearly dominate the large wave-numbers irrespective of the other parameters, so as we might expect, surface tension strongly resists high-curvature perturbations. Likewise, if $\kappa \ll 1$, the linear gravitational term will be largest, so that the long wavelengths will be unstable if $\rho_1 > \rho_2$. If $\rho_2 > \rho_1$, however, it is possible for the flow to be stable for all $k$ and $l$. Once more we can show that the most unstable case has $l = 0$ so we can replace $\kappa$ by $k$. The term in brackets is negative for all $k > 0$, giving stability, if

$$(U_1 - U_2)^2 < \frac{2(\rho_1 + \rho_2)}{\rho_1\rho_2}\left[\gamma g(\rho_2 - \rho_1)\right]^{1/2} \ . \tag{1.17}$$

As $|U_1 - U_2|$ increases, the first wave to go unstable has $k^2 = g(\rho_2 - \rho_1)/\gamma$ We can now calculate the wind speed necessary to drive waves on the surface of a lake. If we put $g = 9.8$, $U_2 = 0$, $\rho_2 = 1000$ $\rho_1 = 1.25$ and $\gamma = 0.074$ appropriate for air above water, we find $U_1 = 6.6\text{m/s}$. At this critical wind-speed, the wavelength of the waves is $2\pi/k = 1.7\text{cm}$.

### Fluids in a rigid cylinder

We have seen that a vortex sheet can be stabilised by the combined action of gravity and surface tension. Is it possible for the gravitational instability of a heavy fluid being held over a light fluid to be nullified? Let us consider a vertical, rigid cylinder of radius $a$. We shall use cylindrical polar coordinates $(r,\, \theta,\, z)$, Once more $z > 0$ contains a fluid of density $\rho_1$ and $z < 0$ has density $\rho_2$. We perturb the interface $z = 0$ to $z = \varepsilon h(r,\, \theta)$.

Then the velocity is $u = \nabla\phi$ with $\nabla^2\phi = 0$, or

$$\mathbf{u} = \left(\frac{\partial\phi}{\partial r},\, \frac{1}{r}\frac{\partial\phi}{\partial\theta},\, \frac{\partial\phi}{\partial z}\right), \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0\ . \tag{1.18}$$

The separable solutions to Laplace's equation in this geometry are

$$\phi = Ae^{\pm kz}J_m(kr)e^{im\theta+st} \tag{1.19}$$

where $J_m$ is a **Bessel function.** $u(r) = J_m(kr)$ is the solution to the problem

$$u'' + \frac{1}{r}u' + u\left(k^2 - \frac{m^2}{r^2}\right) = 0 \qquad \text{with} \quad u(0) = 1 \qquad (1.20).$$

The possible values of $k$ are determined by the rigid wall at $r = a$, where the normal velocity, $\partial\phi/\partial r = 0$ or $J_m'(ka) = 0$. This gives a set of possible values $k = k_{mn}$, for $m = 0,\, 1\ldots,\, n = 1,\, 2\ldots$ Note that $m = 0$ is an axisymmetric disturbance. The kinematic constraint is that $\partial\phi/\partial z = \partial h/\partial t$ on $z = 0$ which suggests we look at perturbations

$$z = \varepsilon h_0 J_m(kr)e^{im\theta+st} \quad\Longrightarrow\quad K = k^2 h\ . \tag{1.21}$$

The pressure boundary condition on $z = h$ is

$$\rho_1\left(\frac{\partial\phi_1}{\partial t} + gh\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + gh\right) + \gamma K\ . \tag{1.22}$$

Putting all this together, we obtain the dispersion relation

$$s^2 = \frac{k[g(\rho_1 - \rho_2) - \gamma k^2]}{\rho_1 + \rho_2} \qquad \text{where} \quad J_m'(ka) = 0. \tag{1.23}$$

We see that we get stability provided the smallest admissable value of $ka$ satisfies

$$(ka)^2_{\mathbf{min}} > \frac{ga^2(\rho_1 - \rho_2)}{\gamma}\ . \tag{1.24}$$

It turns out that the smallest value of $ka$ occurs when $m = 1$, and $(ka)_{\mathbf{min}} \simeq 1.8$. This predicts that the largest pipe radius for which surface tension can support water above air is about 2cm.

Does this agree with your physical intuition, or did you expect the fluid to fall out in an axisymmetric manner?

The theory predicts that if the interface is unstable the air will rush in one side of the pipe (say near $\theta \sim 0$) and the fluid will fall out on the other ($\theta \sim \pi$).

## Inviscid instability of a cylindrical jet

When we turn on a tap a jet of water emerges, whose cylindrical surface quickly becomes kinked and then breaks up into drops. As we shall see, this is a **capillary** or **surface tension** instability. Gravity may be neglected during the evolution of the jet.

Consider a liquid cylinder $0 < r < a$ moving with constant velocity $(0,\, 0,\, U)$ in the axial $z$-direction. The outside we consider to be dynamically negligible, so that the external pressure is constant on the surface. We envisage a surface perturbation of the form

$$r = a(1 + \varepsilon\zeta) \equiv a\left(1 + \varepsilon e^{ikz+im\theta}\right) \ . \tag{1.25}$$

Here $m$ must be an integer, but $k$ can be any positive number. The curvature of this surface $K = \nabla \cdot \widehat{\mathbf{n}}$ takes the form (ignoring terms proportional to $\varepsilon^2$)

$$\widehat{\mathbf{n}} = (1,\, -\varepsilon im\zeta,\, -ika\varepsilon\zeta) \qquad K = \frac{1}{r} + \frac{\varepsilon}{a}(m^2 + k^2a^2)\zeta = \frac{1}{a} + \frac{\varepsilon\zeta}{a}(m^2 + k^2a^2 - 1) \ . \tag{1.26}$$

Taking the velocity $\mathbf{u} = (0, 0, U) + \nabla\phi$, the kinematic boundary condition on $r = a$ is

$$0 = \frac{D}{Dt}(r - a - a\varepsilon\zeta) \quad \Longrightarrow \frac{\partial\phi}{\partial r} = \varepsilon a\zeta(s + ikU) \ , \tag{1.27}$$

and the pressure condition is

$$\rho s\phi + \tfrac{1}{2}\rho 2Uik\phi + \gamma K = \text{constant.} \tag{1.28}$$

The solutions to Laplace's equation take the form

$$\phi = \varepsilon A I_m(kr)\zeta \qquad \text{where} \quad u = I_m(kr) \qquad \text{satisfies} \quad u'' + \frac{u'}{r} - \left(k^2 + \frac{m^2}{r^2}\right) u = 0 \ . \tag{1.29}$$

$I_m$ is called a **modified Bessel function** (compare (1.20).) It behaves like $r^m$ near $r = 0$ and increases monotonically, behaving like $e^{kr}$ as $r \to \infty$.

Combining (1.29), (1.28) and (1.27), we find that

$$(s + ikU)^2 = \frac{\gamma}{\rho a^3}(1 - m^2 - k^2a^2)\left(\frac{kaI_m'(ka)}{I_m(ka)}\right) \ . \tag{1.30}$$

The last factor involving Bessel functions is always positive, so doesn't affect the stability. We see the RHS is negative if $m \geqslant 1$ or if $ka \geqslant 1$, However, the long, axisymmetric waves with $m = 0$ and $0 < ka < 1$ are **unstable.** The jet is unstable to all axial wavelengths larger than its circumference ($2\pi a$). Calculation shows that the greatest value of $s$ occurs at $ka \simeq 0.7$. This was shown by Lord Rayleigh in 1879. Try it in your kitchen! The theory predicts a wavelength of about 4.5 jet diameters.