Civ. Eng. 2 Problem Sheet 5: Vector Calculus

1. Verify the relation

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

for the vector field

$$\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}.$$

(Use the definition $\nabla^2\mathbf{A} = \partial^2\mathbf{A}/\partial x^2 + \partial^2\mathbf{A}/\partial y^2 + \partial^2\mathbf{A}/\partial z^2$).

2. If $\mathbf{A} = 2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\,\mathbf{k}$, $\mathbf{B} = x^2\,\mathbf{i} + yz\,\mathbf{j} - xy\,\mathbf{k}$ and $\phi = 2x^2yz^3$, find:

$$\text{(i) } (\mathbf{A}\cdot\nabla)\phi;\ \ \text{(ii) } \mathbf{A}\cdot(\nabla\phi);\ \ \text{(iii) } (\mathbf{B}\cdot\nabla)\mathbf{A};\ \ \text{(iv) } \mathbf{A}\times(\nabla\phi);\ \ \text{(v) } (\mathbf{A}\times\nabla)\phi.$$

Verify that the answers to (i) & (ii) and (iv) & (v) are the same.

3. Calculate the divergence and curl of the following vector fields:

$$\begin{aligned}
\text{(i) } \mathbf{v} &= \mathbf{r},\ \text{where } \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k};\\
\text{(ii) } \mathbf{v} &= \frac{\mathbf{r}}{|\mathbf{r}|}\ \text{with } \mathbf{r} \text{ as in (i)};\\
\text{(iii) } \mathbf{v} &= \omega\,\mathbf{k}\times\mathbf{r},\ \text{where } \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} \text{ and } \omega \text{ is a constant};\\
\text{(iv) } \mathbf{v} &= \omega\,\mathbf{k}\times\frac{\mathbf{r}}{|\mathbf{r}|},\ \text{with } \mathbf{r} \text{ and } \omega \text{ as in (iii)}.
\end{aligned}$$

4. Establish the identity

$$\nabla\cdot(\phi\mathbf{G}) = \phi\,(\nabla\cdot\mathbf{G}) + (\nabla\phi)\cdot\mathbf{G},$$

for a general scalar field $\phi(x,y,z)$ and general vector field $\mathbf{G} = G_1\,\mathbf{i} + G_2\,\mathbf{j} + G_3\,\mathbf{k}$. Hence evaluate

$$\nabla\cdot\left(\frac{y}{xr^3}\mathbf{r}\right),$$

where $\mathbf{r} = (x,y,z)$ and $r = |\mathbf{r}|$.

5. If $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ is a vector and ϕ is a scalar, prove that

$$\text{(i) } \nabla\times(\nabla\phi) = 0;$$
$$\text{(ii) } \nabla\cdot(\nabla\times\mathbf{A}) = 0.$$

Answers

1. Both right hand side and left hand side are equal to $(2x+2)\,\mathbf{j}$.

2. (i) $8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$; (ii) same as (i); (iii) $(2yz^2 - 2xy^2)\,\mathbf{i} - (x^2yz + 2x^3y)\,\mathbf{j} + (x^2z^2 - 2x^2yz)\,\mathbf{k}$; (iv) $-(6x^4y^2z^2 + 2x^3z^5)\,\mathbf{i} + (4x^2yz^5 - 12x^2y^2z^3)\,\mathbf{j} + (4x^2yz^4 + 4x^3y^2z^3)\,\mathbf{k}$; (v) same as (iv).

3. (i) $\nabla\cdot\mathbf{v} = 3$, $\nabla\times\mathbf{v} = 0$; (ii) $\nabla\cdot\mathbf{v} = 2/r$, $\nabla\times\mathbf{v} = 0$; (iii) $\nabla\cdot\mathbf{v} = 0$, $\nabla\times\mathbf{v} = 2\omega\,\mathbf{k}$; (iv) $\nabla\cdot v = 0$, $\nabla\times\mathbf{v} = (\omega/r)\,k$.

4. $\nabla\cdot(\frac{y}{xr^3}\mathbf{r}) = 0$.