Civ. Eng. 2 Mathematics 2010-11 Solution of PDEs Problem sheet 4

This sheet can be found on the Web: http://www.ma.ic.ac.uk/∼ajm8/Civ2

1. The function $u(x, t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{in} \quad 0 < x < L, \quad t > 0$$

with the boundary conditons

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 \qquad \text{and} \quad u(x, 0) = f(x) \ .$$

Show that its solution is

$$u(x, t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 t/L^2} \ ,$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \ .$$

Hence find the solution when

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \frac{1}{2}L \\ 0 & \text{for} \quad \frac{1}{2}L < x < L \ . \end{cases}$$

What happens as $t \to \infty$? [Ans: $u = \frac{1}{2} + \frac{2}{\pi} \sum_{n\ odd} \frac{1}{n} \sin[\frac{1}{2}n\pi] \cos\frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}$.]

2. (A question from the 1997 exam:) $u(x, y)$ obeys Laplace's equation in a square:

$$u_{xx} + u_{yy} = 0 \qquad \text{in} \quad 0 < x < \pi, \quad 0 < y < \pi \ ,$$

with the boundary conditions

$$u(x, 0) = u(x, \pi) = u(0, y) = 0 \qquad \text{and} \quad u(\pi, y) = \sin^3 y \ .$$

Use the method of separation of variables to find the solution.

[Hint: $\sin 3y = 3\sin y - 4\sin^3 y$.]

[Ans: $u = \frac{3}{4}\frac{\sinh x}{\sinh \pi}\sin y - \frac{1}{4}\frac{\sinh 3x}{\sinh 3\pi}\sin 3y$.]

3. A heat-releasing chemical reaction takes place in a long pipe with insulating walls and fixed temperature at each end. The reaction proceeds at a rate which is proportional to the temperature, so that the temperature distribution $u(x, t)$ obeys the equation

$$u_t = ru + ku_{xx} \qquad \text{in} \quad 0 < x < L \ , \quad t > 0$$

where r and k are known constants. The boundary conditions are

$$u(0,t) = u(L,\,t) = 0 \qquad \text{and} \quad u(x,0) = T(x)\ ,$$

for known $T(x)$. Using separation of variables, show that the temperature is given by

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n \exp(\lambda_n t) \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\lambda_n = r - \frac{n^2\pi^2 k}{L^2} \qquad \text{and} \quad b_n = \frac{2}{L}\int_0^L T(x)\sin\left(\frac{n\pi x}{L}\right)dx\ .$$

Under what conditions will the temperature increase without limit, leading to an explosion? [Ans: Meltdown if $r > \pi^2 k/L^2$.]

4. (From the 2009 exam:) A clarinet can be considered as a vibrating thin pipe of length L, open at one end and closed at the other. Air flow in a clarinet is governed by the equation for $u(x,t)$

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

where c is the constant sound speed, t is time and x denotes distance from the closed end. The boundary conditions on u are

$$u(0,t) = 0 \qquad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

Use the method of separation of variables to find the general solution for u.

The *vibration frequencies* are the positive values of ω such that a possible solution is $u = f(x)\cos(\omega t)$. Show that the second smallest vibration frequency is 3 times the smallest.

Find the particular solution satisfying the initial conditions

$$u(x,0) = 0 \qquad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{3x}{2L}\right).$$

5. Show that the partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic and transform the equation to the canonical form $u_{\xi\eta} = 0$. Obtain the general solution of the transformed equation in terms of two arbitrary functions. Deduce the general solution of the original equation and find the particular solution which satisfies the following boundary conditions at $y = 0$:

$$u = 0, \qquad \frac{\partial u}{\partial y} = 2x\exp(-x^2) \qquad \text{for all } x.$$

$\left[\text{Answer: } u = \frac{1}{3}\left(e^{-(x-2y)^2} - e^{-(x+y)^2}\right).\right]$