Civ. Eng. 2 Maths 2010-11 Problem Sheet 1: Ordinary Differential Equations

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/Civ2

1. Show that x^2 is a solution of the homogeneous differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 0.$$

Hence find the general solution to the inhomogeneous equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 1,$$

by seeking a solution of the form $y = x^2u(x)$ and deriving and solving the ode for u.

2. Show that $y = \cos x$ is a solution of

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + \left(\sec^2 x\right)\, y = 0,$$

and hence find the general solution to the equation

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + \left(\sec^2 x\right)\, y = \cos x.$$

Find also the particular solution that satisfies the boundary conditions

$$y = 1 \text{ when } x = 0, \quad y = \tfrac{1}{2}\left(1 + \ln 2\right) \text{ when } x = \tfrac{1}{3}\pi.$$

3. Find the value of λ such that $y = x^\lambda$ is a solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - \left(\frac{6}{x^2} + \frac{2}{x}\right)y = 0.$$

4. Show that the substitution $x = t^2$ transforms the ode

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \qquad (*)$$

into the equation

$$\frac{d^2y}{dt^2} + y = 0.$$

Hence find the general solution of equation (*).

5. Find the general solution of

$$\frac{d^2y}{dx^2} - (\coth x)\,\frac{dy}{dx} + (4\sinh^2 x)y = 0$$

by making the substitution $t = \cosh x$ to transform it to a simpler form.

6. Use the substitution $x = \sin t$ to determine the general solution (valid for $-1 \leqslant x \leqslant 1$) of

$$(1 - x^2)\frac{d^2y}{dx^2} + \left(2(1 - x^2)^{1/2} - x\right)\frac{dy}{dx} + y = \sin^{-1}(x).$$

Answers

1. $y = (1/4)x^2 \ln x + Ax^2 + B/x^2$.

2. GS: $y = A\cos x + B\cos x \ln(\sec x + \tan x) - \cos x \ln(\cos x)$; PS: $y = (1 - \ln(\cos x))\cos x$.

3. $\lambda = -2$.

4. $y = A\cos\left(\sqrt{x}\right) + B\sin\left(\sqrt{x}\right)$.

5. $y = A\cos\left(2\cosh x\right) + B\sin\left(2\cosh x\right)$.

6. $y = A\exp(-\sin^{-1}(x)) + B\sin^{-1}(x)\exp(-\sin^{-1}(x)) - 2 + \sin^{-1}(x)$.