Civil Eng. 2 Mathematics **Solutions to sheet 4 on PDEs**

This sheet can be found on the Web: http://www.ma.ic.ac.uk/∼ajm8/Civ2

1. Following the arguments in lectures, we look for a solution to the PDE of the form $u(x,\,t) = X(x)T(t)$, for some functions X and T. Then

$$u_t = u_{xx} \quad\Longrightarrow\quad X(x)T'(t) = X''(x)T(t) \quad\Longrightarrow\quad \frac{T'}{T} = \frac{X''}{X} = -p^2\ ,$$

as the last equation states a function of t is equal to a function of x, and hence both functions must be constant. We choose a negative constant to enable us to satisfy the boundary conditions at $x = 0,\ L$. Solving the above equations, we have

$$T = C\exp(-p^2 t) \qquad \text{and} \quad X = A\cos px + B\sin px\ ,$$

where A, B and C are constants. If we insist that $X'(0) = X'(L) = 0$, then since $X' = -Ap\sin px + Bp\cos px$, we have

$$B = 0 \qquad \text{and} \quad Ap\sin pL = 0 \quad\Longrightarrow\quad p = n\pi/L \qquad \text{where} \quad n = 0,\,1,\,2,\,3\ldots$$

The separable solutions are therefore $u = A_n\cos(n\pi x/L)\exp(-n^2\pi^2 t/L^2)$. An arbitrary linear sum of these separable solutions will still satisfy both the PDE and the boundary conditions at $x = 0, L$, so that we can write

$$u = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 t/L^2}\ .$$

We now consider the initial condition $u(x,\,0) = f(x)$, which requires that

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\ .$$

Multiplying both sides of the equation by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating between 0 and L, using the **orthogonality** relation $\int_0^L \cos(n\pi x/L)\cos(m\pi x/L)dx = 0$ if $m \neq n$, we have

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)\,dx = \int_0^L a_m\cos^2\left(\frac{m\pi x}{L}\right)\,dx = \tfrac{1}{2}La_m\ ,$$

which determines the constants a_n. With the given $f(x)$, we have

$$a_0 = \frac{2}{L}\int_0^{L/2} 1\,dx = 1 \qquad \text{while if } n \neq 0 \quad a_n = \frac{2}{L}\int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{n\pi}\sin\tfrac{1}{2}n\pi\ .$$

Now if $n = 2m+1$, then $\sin(\frac{1}{2}n\pi) = (-1)^m$, while it is zero if n is even. Thus

$$u(x,\,t) = \tfrac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos\left(\frac{(2m+1)\pi x}{L}\right) \exp\left[-(2m+1)^2 \frac{\pi^2 t}{L^2}\right] .$$

As $t \to \infty$, the transients die away and $u \to \frac{1}{2}$, which is the average amount of heat in the system at $t = 0$. No heat can escape because of the insulating boundary conditions.

2. Let $u(x,\,y) = X(x)Y(y)$. Then $X''/X = -Y''/Y =$ constant. There are two homogeneous boundary conditions in the y-direction, (i.e. there are two values of y at which $u = 0$), suggesting that we use sines in the y-direction and exponentials in the x-direction. The solutions for $Y(y)$ which are zero at $y = 0,\, \pi$ are $Y = B \sin ny$, where n is an integer. Then $X'' = n^2 X$, and the solutions to that which are zero at $x = 0$ are $X(x) = C \sinh nx$. Taking an arbitrary sum of these gives

$$u(x,\,y) = \sum_{n=1}^{\infty} B_n \sinh nx \sin ny \qquad \text{so that} \quad u(\pi,\,y) = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin ny \ .$$

Comparing with the given data, $u(\pi,\,y) = \frac{3}{4}\sin y - \frac{1}{4}\sin 3y$, we see that

$$u(x,\,y) = \tfrac{3}{4}\frac{\sinh x}{\sinh \pi}\sin y - \tfrac{1}{4}\frac{\sinh 3x}{\sinh 3\pi}\sin 3y \ .$$

3. Once more, seek solutions $u(x,\,t) = X(x)T(t)$. The equation gives

$$XT' = rXT + kX''T \qquad \text{or} \qquad \frac{T'}{T} = r + k\frac{X''}{X} = \text{constant},$$

by the usual argument. Therefore X''/X must be constant, and the boundary conditions require that $X = B\sin(n\pi x/L)$, as before. Therefore

$$\frac{T'}{T} = r - \frac{kn^2\pi^2}{L^2} = \lambda_n \qquad \text{so that} \quad T = Ce^{\lambda_n t} \ .$$

Taking a general linear sum of the separable solutions gives

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n \exp(\lambda_n t) \sin\left(\frac{n\pi x}{L}\right) \tag{$\dagger$}$$

and the initial condition $u(x,\,0) = U(x)$ requires, as previously, that

$$b_n = \frac{2}{L}\int_0^L U(x)\sin\left(\frac{n\pi x}{L}\right) dx \ .$$

Looking at (†), it is clear that the behaviour for large time depends on the signs of λ_n. If all of these values are negative, then $u \to 0$ as $t \to \infty$, and the reaction is stable. If however $\lambda_n > 0$ for at least one value of n, then the temperature will increase without limit, and we will have a catastrophic meltdown, or explosion. From the form of λ_n, we see that $\lambda_1 > \lambda_2 > \lambda_3 \ldots$, so that this will occur if and only if $\lambda_1 > 0$, i.e. if $r > \pi^2 k/L^2$.

4. The standard separation of variables technique gives $u = f(x)g(t)$ where, for some value of k, $f = A\cos kx + B\sin kx$ and $g = C\cos kct + D\sin kct$. Imposing $u = 0$ at $x = 0$ we have $C = 0$. Imposing $u_x = 0$ at $x = L$ gives

$$0 = f'(L) = kB\cos(kL) \qquad \Longrightarrow \qquad kL = \tfrac{1}{2}\pi + n\pi \qquad \text{for} \quad n = 0, 1, 2\ldots$$

Thus the general solution is obtained by an arbitrary sum of thre separable solutions,

$$u = \sum_{n=0}^{\infty} \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)\left(A_n\cos\left((n+\tfrac{1}{2})\frac{\pi ct}{L}\right) + B_n\sin\left((n+\tfrac{1}{2})\frac{\pi ct}{L}\right)\right].$$

The vibration frequencies are therefore $\omega = (n+\frac{1}{2})\pi c/L$ for $n = 0, 1\ldots$ Clearly the smallest is $\pi c/(2L)$ and the second smallest is $3\pi c/(2L)$, which is three times the smallest. So the harmonics on a clarinet are different from those on say a violin.

5. Comparing with $au_{xx} + bu_{xy} + cu_{yy} = f$, we have $a = 2, b = -1, c = -1$. Thus $b^2 - 4ac = 9 > 0$ and the PDE is hyperbolic, as required.

Writing $\xi = x + \beta y$ and $\eta = x + \delta y$, we have $u_x = u_\xi + u_\eta$ and $u_y = \beta u_\xi + \delta u_\eta$. Therefore

$$u_{xy} = \beta u_{\xi\xi} + \delta u_{\eta\eta} + (\beta+\delta)u_{\xi\eta}, \quad u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}, \quad u_{yy} = \beta^2 u_{\xi\xi} + \delta^2 u_{\eta\eta} + 2\beta\delta u_{\xi\eta}.$$

Substituting into the pde, we get

$$(2 - \beta - \beta^2)u_{\xi\xi} + (2 - \delta - \delta^2)u_{\eta\eta} + (4 - 2\beta\delta - \beta - \delta)u_{\xi\eta} = 0.$$

Let β, δ be the roots of $2 - x - x^2 = 0 \quad \Longrightarrow \quad (x+2)(x-1) = 0 \quad \Longrightarrow \quad \beta = -2,\ \delta = 1$. Then the new variables are $\xi = x - 2y$, $\eta = x + y$ and the transformed pde is $9u_{\xi\eta} = 0$.

Integrating partially with respect to ξ, we have for arbitrary functions F and G,

$$\frac{\partial u}{\partial \eta} = F'(\eta) \quad \Longrightarrow \quad u = F(\eta) + G(\xi) = F(x+y) + G(x-2y).$$

Applying the boundary condition $u = 0$ on $y = 0$ gives $0 = F(x) + G(x)$. Thus $G = -F$ and $u = F(x+y) - F(x-2y)$. Then

$$\frac{\partial u}{\partial y} = F'(x+y) + 2F'(x-2y).$$

So on $y = 0$, $u_y = 3F'(x) = 2x\exp(-x^2)$ using the other boundary condition. Integrating, we have $F(x) = \frac{2}{3}\int xe^{-x^2}\,dx = -\frac{1}{3}e^{-x^2} + C$. So finally, the solution is

$$u = \tfrac{1}{3}\Big(\exp[-(x-2y)^2] - \exp[-(x+y)^2]\Big).$$