Civil Eng. 2 Mathematics   Fourier Series   Solutions to sheet 3

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/Civ2

1. $a_n = (1/\pi)\int_0^\pi \cos nx\, dx = 0$ if $n \neq 0$, while $a_0 = 1$. Similarly,

$$b_n = \frac{1}{\pi}\int_0^\pi \sin nx\, dx = \frac{1}{n\pi}[1 - \cos n\pi] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

The required Fourier series is therefore

$$f(x) = \tfrac{1}{2} + \frac{2}{\pi}\sum_{r=1}^{\infty} \frac{\sin[(2r-1)x]}{2r-1} \ .$$

At $x = 0$, all the sines are zero, and the series converges to $\frac{1}{2}$, the average of the values on either side of the discontinuity.

2. $f(x) = x^2$ in $-\pi < x < \pi$. As $f(x)$ is even, $b_n = 0$, while

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx \qquad \text{giving} \quad a_0 = \tfrac{2}{3}\pi^2, \qquad a_n = \frac{4(-1)^n}{n^2} \ .$$

$$\text{Thus} \qquad x^2 = \tfrac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \ . \tag{1}$$

Since $f(x)$ is differentiable and $f(-\pi) = f(\pi)$ we can differentiate the series term by term, to obtain

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} \ .$$

Integrating (1) from, say, 0 to x gives $\frac{1}{3}x^3 = \frac{1}{3}\pi^2 x + 4\sum_{n=1}^{\infty}(-1)^n(\sin nx)/n^3$. We can make this into a Fourier series by substituting for x the series we just calculated, to find

$$x^3 = \sum_{n=1}^{\infty}(-1)^n\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]\sin nx \ .$$

3. Parseval's Theorem states:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(a) When $f(x) = x$ in $-\pi < x < \pi$, we have $a_n = 0$ and $b_n = (-1)^{n+1}/n$. So

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty}(-1)^{2n+2}\frac{1}{n^2} \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \ .$$

(b) When $f(x) = x^2$ in $-\pi < x < \pi$, we have $a_n = (-1)^n \frac{4}{n^2}$ $(n > 0)$ and $a_0 = \frac{2}{3}\pi^2$. So

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \tfrac{2}{9}\pi^4 + \sum_{n=1}^{\infty} \frac{16}{n^4} \qquad \text{or} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{16}\left(\tfrac{2}{5} - \tfrac{2}{9}\right) = \frac{\pi^4}{90}\ .$$

(c) When $f(x) = x^3$ in $-\pi < x < \pi$, we have $a_n = 0$, $b_n = (-1)^n \left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]$. So

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^6 dx = \sum_{n=1}^{\infty}\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]^2 = \sum_{n=1}^{\infty}\left[\frac{144}{n^6} - \frac{48\pi^2}{n^4} + \frac{4\pi^4}{n^2}\right]\ .$$

Using the results from parts (a) and (b), we have

$$\tfrac{2}{7}\pi^6 = 144\sum_{n=1}^{\infty}\frac{1}{n^6} - \tfrac{48}{90}\pi^6 + \tfrac{4}{6}\pi^6 \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{144}\left(\tfrac{2}{7} + \tfrac{48}{90} - \tfrac{4}{6}\right) = \frac{\pi^6}{945}\ .$$

4. Sine series: $1 + x/L = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ where

$$B_n = \frac{2}{L}\int_0^L \left(1 + \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Integrating by parts we find $B_n = \frac{2}{n\pi}(1 - 2\cos n\pi)$.

$$\text{Therefore} \qquad 1 + \frac{x}{L} = \sum_{n=1}^{\infty}\frac{2}{n\pi}(1 - 2(-1)^n)\sin\left(\frac{n\pi x}{L}\right) \qquad \text{for } 0 < x < L.$$

As you saw in lectures, Matlab or Maple are convenient environments for investigating the behaviour of Fourier series and illustrate such things as the Gibbs phenomenon clearly. (This was the systematic overshoot of the series for a discontinuous function.) Of course Matlab can also work out the coefficients.

Incidentally, the Gibbs phenomenon has been rediscovered by various people in different contexts. One such person was Nobel Prize winner Albert Michelson, an extraordinarily careful experimentalist, who measured the speed of light very accurately. (The Michelson-Morley experiment (1887) laid the foundations for the theory of Relativity.)

Michelson built a machine for summing Fourier series, and observed the blip whereby the sum of a finite number of terms always exceeds the values at a discontinuity by the same amount, no matter how many terms of the series are included (his machine used 80). Doubtless this is child's play for civil engineers, but to a mere mathematician this sounds an amazing feat – I suppose he must have used lots of different sized cogs and things…