Civ. Eng. 2 Maths 2009-10 Problem Sheet 1: Solutions

This sheet can be found on the Web: http://www.ma.ic.ac.uk/∼ajm8/Civ2

1. Let $y_1 = x^2$. Then $y_1'' + (1/x)y_1' - (4/x^2)y_1 = 2 + 2 - 4 = 0$, as required.
Now let $y = y_1 u$. Then $y'' = 2u + x^2u'' + 4xu'$, $y' = x^2u' + 2xu$.
So $y'' + (1/x)y' - (4/x^2)y = x^2u'' + 5xu' = 1 \Rightarrow u'' + (5/x)u' = 1/x^2$.
Solve using integrating factor $x^5 \Rightarrow \left(x^5u'\right)' = x^3 \Rightarrow x^5u' = x^4/4 + A \Rightarrow u = (1/4)\ln x + C/x^4 + D$
$\Rightarrow \underline{y = x^2u = (1/4)x^2\ln x + C/x^2 + Dx^2}$.

2. Let $y_1 = \cos x$. Then $y_1'' + \tan x\, y_1' + \sec^2 x\, y_1 = -\cos x - \sin x\tan x + \sec^2 x\cos x$
$= \left(1 - (\cos^2 x + \sin^2 x)\right)/\cos x = 0$, as required.
Let $y = y_1u$. Then $y'' = -u\cos x + u''\cos x - 2u'\sin x,\ y' = u'\cos x - u\sin x$.
Therefore $y'' + \tan x\, y' + \sec^2 x\, y = u''\cos x - u'\sin x = \cos x$.
Integrating factor is $\exp(-\int \tan x\, dx) = \cos x$.
Then $(u'\cos x)' = \cos x \Rightarrow u'\cos x = \sin x + C \Rightarrow u' = \tan x + C\sec x$
$\Rightarrow u = -\ln(\cos x) + C\ln(\sec x + \tan x) + D$
$\Rightarrow y = u\cos x \Rightarrow \underline{y = -\cos x\ln(\cos x) + C\cos x\ln(\sec x + \tan x) + D\cos x}$.
b.c's: $x = 0$, $y = \underline{D = 1}$. $x = \pi/3 : y = -(1/2)\ln(1/2) + (C/2)\ln(2 + \sqrt{3}) + (1/2)$
$= (1/2)(1 + \ln 2) + (C/2)\ln(2 + \sqrt{3}) = (1/2)(1 + \ln 2) \Rightarrow \underline{C = 0}$.
So solution is $\underline{y = (1 - \ln(\cos x))\cos x}$.

3. Let $y_1 = x^\lambda$. Then $y_1'' - y_1' - (6/x^2 + 2/x)y_1 = \lambda(\lambda-1)x^{\lambda-2} - \lambda x^{\lambda-1} - 6x^{\lambda-2} - 2x^{\lambda-1} = 0$.
Equate coefficients of $x^{\lambda-2} \Rightarrow \lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = 3$ or $\lambda = -2$.
Coefficients of $x^{\lambda-1} \Rightarrow \lambda + 2 = 0 \Rightarrow \lambda = -2$. So $\underline{\lambda = -2}$ is the only possible value.

4. If $x = t^2$ then $dy/dt = (dx/dt)\,(dy/dx) = 2t dy/dx = 2x^{1/2}dy/dx$.
Then $d^2y/dt^2 = 2x^{1/2}\frac{d}{dx}\left(2x^{1/2}\frac{dy}{dx}\right) = 4xd^2y/dx^2 + 2dy/dx$.
Therefore the ode becomes $d^2y/dt^2 + y = 0$, as required.
General solution is $y = A\cos t + B\sin t \Rightarrow$ in terms of x, $\underline{y = A\cos(\sqrt{x}) + B\sin(\sqrt{x})}$.

5. If $t = \cosh x$, then $dy/dt = (dx/dt)\,(dy/dx) = (1/\sinh x)(dy/dx)$.
Then $d^2y/dt^2 = (1/\sinh x)\frac{d}{dx}((1/\sinh x)\frac{dy}{dx}) = (1/\sinh^2 x)d^2y/dx^2 - (\cosh x/\sinh^3 x)dy/dx$
$\Rightarrow (\sinh^2 x)d^2y/dt^2 = d^2y/dx^2 - (\coth x)\,dy/dx$.
Therefore the ode becomes $\left(\sinh^2 x\right)d^2y/dt^2 + \left(4\sinh^2 x\right)y = 0 \Rightarrow d^2y/dt^2 + 4y = 0$
$\Rightarrow y = A\cos 2t + B\sin 2t$. In terms of x, the solution is thus $\underline{y = A\cos(2\cosh x) + B\sin(2\cosh x)}$.

6. Let $x = \sin t$. Then $dy/dt = (dx/dt)(dy/dx) = (\cos t)\,dy/dx = (1 - x^2)^{1/2}dy/dx$.
So, $d^2y/dt^2 = (1 - x^2)^{1/2}\frac{d}{dx}((1 - x^2)^{1/2}dy/dx) = (1 - x^2)d^2y/dx^2 - xdy/dx$.
Thus: $(1 - x^2)d^2y/dx^2 - xdy/dx + 2(1 - x^2)^{1/2}dy/dx = d^2y/dt^2 + 2dy/dt$.
The ode therefore becomes $d^2y/dt^2 + 2dy/dt + y = \sin^{-1}(x) = t$.
We have a constant coefficient 2nd order ode. Solve in the normal way by seeking homogeneous (y_H) and particular (y_P) solutions. For y_H look for solution $\propto \exp(\lambda t) \Rightarrow \lambda^2 + 2\lambda + 1 = 0$
$\Rightarrow (\lambda + 1)^2 = 0 \Rightarrow \lambda = -1 \Rightarrow y_H = (A + Bt)\exp(-t)$.
For particular solution try $y_P = Ct + D$.
Substitute into ode to get $2C + Ct + D = t \Rightarrow C = 1, D = -2 \Rightarrow y_P = t - 2$.
So the general solution in terms of t is $y = y_H + y_P = (A + Bt)\exp(-t) + t - 2$.
Writing back in terms of x we have $\underline{y = (A + B\sin^{-1}(x))\exp(-\sin^{-1}(x)) + \sin^{-1}(x) - 2.}$